NON-GENERIC BLOW-UP SOLUTIONS FOR THE CRITICAL FOCUSING NLS IN 1-D.

J. KRIEGER, W. SCHLAG

1. Introduction

We consider the critical focusing NLS in 1-d of the form

$$i\partial_t\psi + \partial_x^2\psi = -|\psi|^4\psi,\ i = \sqrt{-1},\ \psi = \psi(t,x), \tag{1.1}$$

and ψ complex valued. It is well-known that this equation permits standing wave solutions of the form

$$\phi(t,x) = e^{i\alpha t}\phi_0(x,\alpha),\ \alpha > 0$$

Indeed, requiring positivity and evenness in x for $\phi_0(x,\alpha)$ implies for example

$$\phi_0(x,\alpha) = \frac{\alpha^{\frac{1}{2}}(\frac{3}{2})^{\frac{1}{4}}}{\cosh^{\frac{1}{2}}(\frac{\alpha}{2}x)}$$

Another remarkable feature of the equation (1.1) is the large symmetry group carrying solutions into solutions: this is generated by

Galilei transformations:

$$\psi(t,x) \longrightarrow e^{i(\gamma+vx-v^2t)}e^{-i(2tv+\mu)p}\psi(t,x) = e^{i(\gamma+vx-v^2t)}\psi(t,x-2tv-\mu),\ p = -i\frac{d}{dx}$$

$SL(2,\mathbf{R})$-transformations:

$$\psi(t,x) \longrightarrow (a+bt)^{-\frac{1}{2}}e^{\frac{ibx^2}{4(a+bt)}}\psi(\frac{c+dt}{a+bt},\frac{x}{a+bt}),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R})$$

Observe that the latter subsume re-scalings $\psi(t,x) \rightarrow a^{\frac{1}{2}}\psi(a^2t,ax)$ while the former subsume phase-shifts $\psi(t,x) \rightarrow e^{i\gamma}\psi(t,x)$ as well as translations. We usually identify a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R})$ with the corresponding transformation. It is the $SL(2,\mathbf{R})$-transformations that distinguish the critical NLS from the sub- and supercritical NLS, and allows us to exhibit explicit blow-up solutions: indeed, fixing $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R})$, we have the explicit solution

$$f(t,x) = (a+bt)^{-\frac{1}{2}}e^{i\frac{c+dt}{a+bt}}e^{\frac{ibx^2}{4(a+bt)}}\phi_0(\frac{x}{a+bt},1), \tag{1.2}$$

which blows up for $t = -\frac{a}{b}$. Fixing $a \sim 1$, $b \sim -1$, it is then a natural question to ask whether one may perturb the initial data of (1.2) at time $t = 0$ such that the corresponding solution exhibits the same type of blow-up behavior. More precisely, the solution should asymptotically behave like $\sqrt{\frac{1}{T-t}}e^{i\Psi(t,x)}\phi(\frac{x-\mu(t)}{T-t})$ for a bounded function $\mu(t)$ and suitable Schwartz function ϕ, with blow up time T. The recent work of Merle-Raphael [MeRa] has demonstrated that this is generically impossible, i. e. there are open sets of initial

The first author was partially supported by NSF grant DMS-0401177 and the second author by the NSF grant DMS-0300081 and a Sloan fellowship.

data containing $f(0,x)$ in their closure[1] and such that their blow-up behavior is of the following type, which we henceforth refer to as 'generic':

$$\psi(t,x) \sim e^{i\mu(t)}\lambda^{\frac{1}{2}}(t)\phi(\lambda(t)x),\ \lambda(t) \sim \sqrt{\frac{\log|\log(T-t)|}{T-t}}$$

Blow-up solutions of this type were first constructed in a remarkable paper by G. Perelman [Per2], but for non-generic initial data sets. This blow-up rate was shown to be stable in [Ra]. Moreover, in [MeRa] the authors showed that for initial data in a sufficiently small neighborhood of $f(0,x)$ the only possible blow-up speeds are the generic speed or else at least as fast as the explicit speed; we now refer to the latter as 'non-generic'. The issue remains as to whether perturbations of the initial data $f(0,x)$ in certain directions would result in the non-generic blow-up type. The first and to our knowledge only result of this type was established by Bourgain-Wang [BW][2], and asserts the following:

Theorem 1.1. *[Bourgain-Wang] Let z_ϕ be the local-in-time solution of*

$$i\psi_t + \triangle\psi + |\psi|^4\psi = 0,\ \psi(0) = \phi,$$

which for smooth ϕ exists on an interval $[-\delta,\delta]$ for $\delta = \delta(\phi)$ small enough. Then provided ϕ is smooth and vanishes sufficiently fast at 0, i.e. $|\phi(x)| \lesssim |x|^A$ for A large enough[3], there exists smooth $w(t,x)$ in a suitable function space with $w(0,x) = 0$ and such that

$$\psi(t,x) = t^{-\frac{1}{2}}e^{\frac{x^2-4}{4it}}\phi_0(\frac{x}{t},1) + z_\phi(t,x) + w(t,x) \tag{1.3}$$

solves (1.1) on $[-\delta,0]$. One may let $\delta \to \infty$ by letting $\phi \to 0$.

The key behind this result is to first undo the blow-up by applying a pseudo-conformal transformation C^{-1} where $C\psi(t,x) = t^{-\frac{1}{2}}e^{\frac{x^2}{4it}}\psi(\frac{x}{t},-\frac{1}{t})$ and then employ the properties of the linear evolution associated with the linearization around the standing wave $e^{it}\phi_0(x)$. More precisely, one passes to the vector valued function $\begin{pmatrix} \psi(t,x) \\ \overline{\psi(t,x)} \end{pmatrix}$ and observes that if $\psi(t,x) = e^{it}(\phi_0(x,1) + u(t,x))$ solves (1.1) then we have

$$(i\partial_t + \mathcal{H})\begin{pmatrix} u(t,x) \\ \overline{u(t,x)} \end{pmatrix} = N(u)$$

where we put

$$\mathcal{H} = \begin{pmatrix} \partial_x^2 - 1 + 3\phi_0^4(x,1) & 2\phi_0^4(x,1) \\ -2\phi_0^4(x,1) & -\partial_x^2 + 1 - 3\phi_0^4(x,1) \end{pmatrix} \tag{1.4}$$

and $N(u)$ is of order ≥ 2 in u. The spectral properties of the operator $\mathcal{H}$ are well-known after the pioneering work of Weinstein [Wei1] as well as Buslaev-Perelman [BusPer] and Perelman [Per2]. In particular, the linear equation $(i\partial_t + \mathcal{H})\begin{pmatrix} u \\ \bar{u} \end{pmatrix} = 0$ only displays algebraic instabilities. More precisely, the spectrum of $\mathcal{H}$ has essential part $(-\infty,-1] \cup [1,\infty)$ and discrete spectrum $\{0\}$ of geometric multiplicity 2 and algebraic multiplicity 6. A solution $\Phi(x) = \begin{pmatrix} \phi(x) \\ \bar{\phi}(x) \end{pmatrix}$ in the generalized root space satisfies for example $||e^{it\mathcal{H}}\Phi(x)||_{L_x^2} \lesssim (1+t^3)\int_{-\infty}^{\infty} e^{-c|x|}|\Phi(x)|dx$. In order to counteract this growth behavior at infinity, Bourgain and Wang use the ansatz $u(t,x) = C^{-1}z_\phi e^{-it} + \tilde{w}(t,x)$, see (1.3), and then observe that the non-linearity of the resulting equation for $\begin{pmatrix} \tilde{w} \\ \bar{\tilde{w}} \end{pmatrix}$ decays sufficiently rapidly at infinity (due to the local decay of $C^{-1}z_\phi e^{-it}$) that it overwhelms any losses due to the algebraic instability of $\mathcal{H}$. The fact that the 'static coupling' (1.3) barely exploits the symmetries of the equation and in particular doesn't allow the standing wave to 'drift'

[1] With respect to any reasonable norm.

[2] The authors state this Theorem for the case of $d = 1, 2$ dimensions.

[3] The numerology in [BW] appears to imply $A \geq 16$

certainly implies the sub-optimality[4] of Theorem 1.1.
Indeed, a careful analysis of the root-space of $\mathcal{H}$ as in [Wei1] shows that 5 of the generalized root modes (the 'good modes') 'are due to' the internal symmetries of (1.1), while there is one 'exotic mode', see the ensuing discussion. This intimates that upon applying suitable internal symmetries to the standing wave $e^{it}\phi_0(x,1)$ in time-dependent fashion (i. e. using a modulation-theoretic approach), one should be able to control the root part of the radiation corresponding to the good modes, and indeed obtain a co-dimension 1 stable manifold of initial data (due to the 'exotic mode' which cannot be so controlled) resulting in the non-generic blow-up profile:

Conjecture 1.2. *[Galina Perelman] There exists a co-dimension 1 manifold of initial data resulting in the non-generic blow-up behavior.*

We note that this is also implicitly mentioned although in less precise form in [B].
This falls in neatly with recent results in [Sch] and [KriSch1], the latter closely following the former, which in the context of the L^2-super-critical NLS (the cubic in 3-d in [Sch] and the full super-critical range in 1-d in [KriSch1]) established existence of co-dimension 1 manifolds of initial data resulting in globally (for $t\to+\infty$) smooth solutions. The co-dimension 1 here has to do with one exponentially unstable mode (in the forward time direction); the generalized root space has only dimension 4, in one-one correspondence with the internal symmetries.

However, the critical case differs in a pivotal aspect from the super-critical one: this is due to the nature of the dynamics governing the 'exotic mode'. Recall that this mode, together with one of the 'geometric modes', turn into the exponentially growing and decaying modes corresponding to the purely-imaginary eigenvalues in the super-critical case. The latter are controlled by a hyperbolic ODE. As observed in [Sch] and later [KriSch1], this hyperbolic ODE produces good decay behavior at infinity, provided this is coupled with suitable initial conditions. In particular, the linear theory in [KriSch1] allows one to get by with rather weak local decay estimates for the dispersive part of the radiation, such as $|\phi U_{dis}(t,.)|\lesssim\langle t\rangle^{-1-\epsilon}$ for a Schwartz function ϕ.
In the critical case, the imaginary eigenvalues merge at zero. The evolution of the 'exotic root mode' which is not due to internal symmetries is then controlled by a Riccati type ODE, (3.40). It is then clear that the linear estimates alone derived in [KriSch1] are nowhere near enough to control the evolution of this mode, in a fashion similar to [Sch] or [KriSch1]. Instead, we have to exploit subtle null-structures in the nonlinearity, i. e. exploit the oscillatory behavior of the radiation part, to eke out additional decay in time. Such estimates, while similar in nature to null-form estimates for nonlinear wave equations and the defocusing NLS, appear completely new in this context, especially since we don't work with the ordinary Fourier transform but with the *distorted Fourier transform*. In particular, the consideration in section 5.1, which produces almost optimal local decay for the dispersive radiation part, appear of interest in their own right, and might be useful in other contexts. The crucial observation in this section is the fact that the expression $|U|^2$ has a certain smoothing property.

In this paper we prove the following version of Conjecture 1.2: let[5]

$$\mathcal{T}_\infty=e^{-i(v_\infty^2 s+\gamma_\infty+v_\infty y)}e^{i(2v_\infty s+y_\infty)p}\begin{pmatrix} a_\infty & b_\infty\\ 0 & a_\infty^{-1}\end{pmatrix}$$

Also, let the generalized root space of $\mathcal{H}$ be generated by the (vector-valued) Schwartz functions $\eta_{i,\text{proper}}=\begin{pmatrix}\eta^1_{i,\text{proper}}\\ \eta^1_{i,\text{proper}}\end{pmatrix}$, $i=1,\dots,6$, and that of $\mathcal{H}^*$ be generated by the Schwartz functions $\xi_{i,\text{proper}}(x)$, $i=1,\dots,6$, viz. the ensuing discussion.

---

[4]In the sense that the set of initial data resulting in the non-generic blow-up should be significantly larger than indicated there.

[5]We identify the matrix with its associated transformation.

Our main result is the following

Theorem 1.3. *Fix real parameters $\lambda \sim 1, \beta \sim 1, \omega \lesssim 1, \gamma \lesssim 1, \mu \lesssim 1$. Given a vector valued function $x \longrightarrow \begin{pmatrix} U \\ \bar{U} \end{pmatrix}(0,x)$ satisfying $\langle \begin{pmatrix} U \\ \bar{U} \end{pmatrix}(0,.), \xi_{i,proper}\rangle = 0 \forall i$, as well as the smallness condition $|||U(0,.)||| < \delta$ for a suitable norm*[6] *$|||.|||$ and sufficiently small $\delta > 0$, there exist numbers $\tilde{\lambda}_i \in \mathbf{R}$ with $|\tilde{\lambda}_i| \lesssim |||U(0,.)|||^2$ and parameters $\{a_\infty, b_\infty, v_\infty, y_\infty, \gamma_\infty\}$ with $|a_\infty - \lambda| \lesssim |||U(0,.)|||^2$, $|b_\infty - \beta\lambda| \lesssim |||U(0,.)|||^2$, $|v_\infty - \frac{\beta\lambda\mu}{2} - \omega| \lesssim |||U(0,.)|||^2$, $|y_\infty - \lambda\mu| \lesssim |||U(0,.)|||^2$, $\gamma_\infty = \gamma_\infty(,\gamma,\lambda,\beta,\omega,\mu) + O(|||U(0,x)|||^2)$, such that the initial data*

$$\psi(0,x) := W(0,x) + \mathcal{T}_\infty^{-1}[U(0,x) + \sum_{i=1}^{6} \tilde{\lambda}_i \eta^1_{i,proper}]$$

lead to solutions of (1.1) *blowing up in finite time according to the non-generic profile, where*

$$W(0,x) = e^{i(\gamma+\omega(x-\mu))} e^{-i\frac{\beta}{4}\lambda^2(x-\mu)^2} \sqrt{\lambda}\phi_0(\lambda(x-\mu),1)$$

More precisely, the solution decouples as

$$\psi(t,x) = e^{i(\gamma(t)+\omega(t)(x-\mu(t)))} e^{-i\frac{\beta(t)}{4}\lambda^2(t)(x-\mu(t))^2} \sqrt{\lambda(t)}\phi_0(\lambda(t)(x-\mu(t)),1) + R(t,x),$$

where $\lambda(t) \sim \frac{1}{T-t}$ for a suitable $T > 0$, and $\mu(t)$ is bounded, while we have the bounds

$$\sup_{0\le t<T} ||R(t,x)||_{L_t^\infty} \lesssim \delta,\ ||R(t,x)||_{H^1} \lesssim \delta(T-t)^{-1},\ 0 \le t < T$$

In particular, $||\psi(t,x)||_{H^1} \sim (T-t)^{-1}$ for $0 \le t < T$.

Remark: We observe that this result would imply Conjecture 1.2 if we could show Lipschitz continuous dependence of the $\tilde{\lambda}_i$, a_∞ etc on $U(0,x)$, along the lines in [KriSch1]. However, we cannot establish this. Indeed, even demonstrating the possibility or impossibility of *choosing these parameters in continuous fashion* appears extremely difficult.

We now outline the strategy used to prove this Theorem: there are the following four stages:

Stage A: Setting up the equations for radiation part and modulation parameters.

Instead of the static coupling (1.3), we make the ansatz

$$\psi(t,x) = W(t,x) + R(t,x),\ W(t,x) = e^{i(\theta(t,x))} e^{-i\frac{\beta}{4}(t)\lambda^2(x-\mu(t))^2} \sqrt{\lambda(t)}\phi_0(\lambda(t)(x-\mu(t)),1) \tag{1.5}$$

In order to ensure that this solution behaves like a non-generic blow-up solution, we impose the condition $\lambda(t) \sim \frac{1}{t_*-t}$ for suitable $t_* \in \mathbf{R}_{>0}$. We shall similarly have to carefully specify the 'asymptotic behavior' of the remaining parameters as we approach blow-up time. In order to specify the evolution of these parameters, we impose suitable orthogonality conditions: letting $\xi_{i,\text{proper}}, i = 1, \dots, 6$ denote a certain basis of the generalized root space of $\mathcal{H}^*$ (recall (1.4)) to be specified below, we impose[7]

$$\langle \begin{pmatrix} R \\ \bar{R} \end{pmatrix}, \xi_i\rangle = 0,\ \xi_i = \begin{pmatrix} e^{i\Psi(t,x)} & 0 \\ 0 & e^{-i\Psi(t,x)} \end{pmatrix} \sqrt{\lambda(t)}\xi_{i,\text{proper}}(\lambda(t)(x-\mu(t))), i = 2, \dots, 6 \tag{1.6}$$

where we have introduced the notation

$$\Psi(t,x) = \theta(t,x) - \frac{\beta(t)}{4}\lambda^2(t)(x-\mu(t))^2$$

For later reference , we define η_i correspondingly, with $\xi_{i,\text{proper}}$ replaced by $\eta_{i,\text{proper}}$. This is analogous to the procedure in [Sch], [KriSch1], where the generalized root space is only 4-dimensional. The above orthogonality condition then implies that at time t the radiation part *when projected onto the generalized root space of the instantaneous linearization around the drifting soliton* gives zero. Note that the fact the we no longer work

[6]see definition 4.4

[7]The root functions $\xi_{i,\text{proper}}$, $i = 1, \dots, 5$ here are chosen to be the 'good modes' in one-one relation with the internal symmetries, while the root function $\xi_{6,\text{proper}}$ is the 'exotic mode' due to the degeneracy in the critical case.

with a static standing wave forces us to work with modifications of the operator $\mathcal{H}$.

Instead of working with the formulation (1.5), we then revert to a 'different Gauge' as in [BW]. Specifically, we apply a suitable transformation $\mathcal{T}_\infty$,

$$\mathcal{T}_\infty = e^{-i(v_\infty^2 s+\gamma_\infty+v_\infty x)}e^{i(2v_\infty s+y_\infty)p}\begin{pmatrix} a_\infty & b_\infty \\ 0 & a_\infty^{-1}\end{pmatrix}, \, p=-i\frac{d}{dx}, \tag{1.7}$$

to $\psi(t,x)$ which is to undo the singular behavior and should map the blow-up time t_* to $s=\infty$. To see how the 'coefficients at infinity' a_∞ etc. should be chosen, we observe that

$$(\mathcal{T}_\infty F)(s,y) = e^{-i\tilde{\Psi}(s,y)}\lambda_\infty^{-\frac{1}{2}}(s)F(\int_0^s \lambda_\infty^{-2}(\sigma)d\sigma, \lambda_\infty^{-1}(s)y+\mu_\infty(s))$$

where

$$\lambda_\infty(s)=a_\infty+b_\infty s,\, \mu_\infty(s)=\frac{2v_\infty s+y_\infty}{a_\infty+b_\infty s},\, \tilde{\Psi}_\infty(s,y)=v_\infty^2 s+\gamma_\infty+v_\infty y-\frac{b_\infty(y+2v_\infty s+y_\infty)^2}{4(a_\infty+b_\infty s)},$$

and therefore

$$\begin{aligned}&e^{-is}(\mathcal{T}_\infty W)(s,y)\\ &=e^{-i(s+\tilde{\Psi}_\infty(s,y))+i\Psi(t(s),\mu_\infty(s)+\lambda_\infty^{-1}(s)y)}\lambda_\infty^{-\frac{1}{2}}(s)\lambda^{\frac{1}{2}}(t(s))\phi_0(\lambda(t(s))(\mu_\infty(s)-\mu(t(s))+\lambda_\infty^{-1}(s)y),1),\end{aligned}$$

where we have put $t(s)=\int_0^s\lambda_\infty^{-2}(\sigma)d\sigma=\frac{a_\infty^{-1}s}{a_\infty+b_\infty s}$. The above suggests that we should impose $\lambda(t(s))\sim\lambda_\infty(s)$, $\mu_\infty(s)\sim\mu(t(s))$ as $s\to\infty$ in a precise sense to be specified. In particular, we have $t_*=\frac{1}{a_\infty b_\infty}$ for the blow-up time.

We shall now work with the vector valued function

$$\begin{pmatrix} U\\ \bar{U}\end{pmatrix} := \mathcal{M}\mathcal{T}_\infty\begin{pmatrix} R\\ \bar{R}\end{pmatrix},\, \mathcal{M}=\begin{pmatrix} e^{-is} & 0\\ 0 & e^{is}\end{pmatrix}$$

Then introduce the functions $\tilde{\eta}_i=\mathcal{M}\mathcal{T}_\infty\eta_i$, $\tilde{\xi}_i=\mathcal{M}\mathcal{T}_\infty\xi_i$. One deduces the following equation for $\begin{pmatrix} U\\ \bar{U}\end{pmatrix}$:

$$i\partial_s\begin{pmatrix} U\\ \bar{U}\end{pmatrix}+\mathcal{H}(s)\begin{pmatrix} U\\ \bar{U}\end{pmatrix} = -i(\dot{\lambda}\lambda^{-1}-\beta\nu^2)(\tilde{\eta}_2-\beta\tilde{\eta}_5/2+\omega\tilde{\eta}_4) \tag{1.8}$$

$$+\frac{i}{4}(\dot{\beta}+\beta^2\nu^2)\tilde{\eta}_5+i(\nu^2-\dot{\gamma}+\nu^2\omega^2)\tilde{\eta}_1 \tag{1.9}$$

$$-i(\dot{\omega}+\beta\omega\nu^2)\tilde{\eta}_4-i\nu(\dot{\mu}\lambda_\infty-2\nu\omega)(-\omega\tilde{\eta}_1-\tilde{\eta}_3+\beta\tilde{\eta}_4/2)+N(U,\pi), \tag{1.10}$$

where we use $\nu(s)=\frac{\lambda(t(s))}{\lambda_\infty(s)}$, $\lambda(t(s))=\lambda(s)$, $\dot{\lambda}=\frac{\partial}{\partial s}\lambda(s)$, and $N(U,\pi)$ is quadratic in U but also depends on the modulation parameters $\lambda(s)$ etc., as well as the parameters at infinity a_∞ etc. We denote the latter collectively as π, following the notation in [Sch], [KriSch1]. The operator $\mathcal{H}(s)$ in the preceding is given by

$$\mathcal{H}(s):=\begin{pmatrix} \partial_y^2-1+3\nu^2(s)\phi_0^4(\lambda(\mu_\infty-\mu+\lambda_\infty^{-1}y)) & 2\nu^2\phi_0^4(\lambda(\mu_\infty-\mu+\lambda_\infty^{-1}y))e^{2i(\Psi-\Psi_\infty)}\\ -2\nu^2\phi_0^4(\lambda(\mu_\infty-\mu+\lambda_\infty^{-1}y))e^{-2i(\Psi-\Psi_\infty)} & -\partial_y^2+1-3\nu^2(s)\phi_0^4(\lambda(\mu_\infty-\mu+\lambda_\infty^{-1}y))\end{pmatrix},$$

where we use the notation $\Psi_\infty(s,y):=\tilde{\Psi}_\infty(s,y)+s$. The orthogonality relations (1.6) become the following:

$$\langle\begin{pmatrix} U\\ \bar{U}\end{pmatrix},\tilde{\xi}_i\rangle=0,\, i=2,\dots,6, \tag{1.11}$$

and upon differentiating with respect to s imply a set of ODE's for the parameters $\lambda(s)$ etc. The crux now is to deduce a priori estimates for the transformed radiation part $\begin{pmatrix} U\\ \bar{U}\end{pmatrix}$ as well as for the modulation parameters;

the latter need to satisfy the required asymptotic estimates for $s \to \infty$. In order to control the radiation part, one essentially[8] invokes a decomposition

$$\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(s,y) = \begin{pmatrix} U \\ \bar{U} \end{pmatrix}_{dis}(s,y) + \sum_{i=1}^{6} \lambda_i(s)\eta_{i,\text{proper}}(y),$$

where the coefficients $\lambda_i(s)$, $i = 1, \ldots, 5$, are determined by the orthogonality relations (1.11). The coefficient $\lambda_6(s)$ is determined by means of the requirement $\lim_{s\to\infty} \lambda_6(s) = 0$, which forces an initial condition $\lambda_6(0)$, similarly to the super-critical case treated in [Sch], [KriSch1]. By comparison to the latter, though, controlling $\lambda_6(s)$ appears more difficult, and requires the development of rather new technology. Specifically, a careful analysis of the modulation equations reveals that one needs to control quantities of the form $\int_T^\infty t\lambda_6(t)dt$, which upon substituting the solution for the ODE satisfied by $\lambda_6(t)$ results in quadratic expressions of at worst the form[9] $\int_T^\infty t\int_t^\infty \langle U^2(s) - \bar{U}^2(s), \phi\rangle ds dt$ etc., where ϕ stands for a suitable Schwartz function. This shows that one should aim for a local decay of the radiation part of at least the strength $|\langle U(t,.), \phi\rangle| \lesssim \langle t\rangle^{-\frac{3}{2}}$ in order to be able to estimate this expression; indeed, this local decay rate is in accordance with the linear estimates derived in [KriSch1]. However, we are dealing with a nonlinear problem here. This is the first significant difficulty to be overcome:

Stage B: Deducing the strong local dispersive[10] estimate for the radiation part.

Schematically, the equation (1.8) etc can be recast as

$$(i\partial_s + \mathcal{H})\begin{pmatrix} U \\ \bar{U} \end{pmatrix} = VU + \begin{pmatrix} |U|^4U \\ -|U|^4\bar{U} \end{pmatrix},$$

where $\mathcal{H} = \begin{pmatrix} \partial_y^2 - 1 + 3\phi_0^4(.,1) & 2\phi_0^4(.,1) \\ -2\phi_0^4(.,1) & -\partial_y^2 + 1 - 3\phi_0^4(.,1) \end{pmatrix}$, and V, a Schwartz function, depends on U as well as the modulation parameters etc. The local[11] expression VU is due to interactions of the drifting soliton with itself as well as to interactions of the radiation with the drifting soliton, while the non-local quintilinear expressions $|U|^4U$ come from interactions of the radiation part with itself. While the root part of $\begin{pmatrix} U \\ \bar{U} \end{pmatrix}$ is controlled in terms of the coefficients $\lambda_i(s)$, $i = 1, \ldots, 6$, whose estimation is relegated to the third stage, the dispersive part[12] (viz. the next section for the linear background) $\begin{pmatrix} U \\ \bar{U} \end{pmatrix}_{dis}$ satisfies

$$(i\partial_s + \mathcal{H})\begin{pmatrix} U \\ \bar{U} \end{pmatrix}_{dis} = [VU + \begin{pmatrix} |U|^4U \\ -|U|^4\bar{U} \end{pmatrix}]_{dis},$$

The really difficult contribution on the right comes from the non-local quintilinear term: note that the standard way to deduce the local estimate is to combine the linear estimate[13]

$$|\langle e^{it\mathcal{H}}\phi_{dis}, \psi\rangle| \lesssim t^{-\frac{3}{2}}||\langle x\rangle\phi||_{L_x^1}||\langle x\rangle\psi||_{L_x^1}$$

with Duhamel's formula, which then forces us to estimate the expression

$$\int_0^t \langle t-s\rangle^{-\frac{3}{2}}||\langle x\rangle|U|^4(s,x)U(s,x)||_{L_x^1}ds \tag{1.12}$$

[8]For technical reasons, one uses such a decomposition for a slightly transformed function $\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}$.

[9]We are again careless here; the expression should really involve $\tilde{U}^2 - \bar{\tilde{U}}^2$.

[10]More precisely, we establish the strong local dispersive estimate up to an arbitrarily small error.

[11]We refer to expressions which are Schwartz functions alternatively as 'local'.

[12]Again, one should really use $\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}$.

[13]We abuse notation here and use letters ϕ, ψ to denote vector valued functions. Also, we let $\langle x\rangle = |x| + 1$.

On the other hand, again from the linear theory summarized in the next section we expect the estimate

$$||\langle x\rangle|U|^4(s,x)U(s,x)||_{L^1_x} \lesssim ||U(s,.)||^3_{L^\infty_x}||xU(s,x)||_{L^2_x}||U(s,x)||_{L^2_x} \lesssim \langle s\rangle^{-\frac{3}{2}}s = s^{-\frac{1}{2}},$$

which only gives the decay $t^{-\frac{1}{2}}$ when substituted into (1.12). One can modify this argument to eke out a local dispersive decay of $\langle t\rangle^{-1+}$, which however is insufficient for controlling the root part and modulation parameters.

The way out of this is to observe that the quintilinear expression exhibits a *special algebraic cancellation structure*, which in combination with the linear theory of $\mathcal{H}$(and in particular the absence of resonances at the edges of the essential spectrum) allows one to significantly improve on the preceding. To explain the use of this algebraic structure heuristically, note that one expects the small-frequency part of $\begin{pmatrix} |U|^4U \\ -|U|^4\bar{U} \end{pmatrix}_{dis}$ to contribute less due to the absence of resonances at the end of the essential spectrum of $\mathcal{H}$. Another reason is that the small frequency part propagates more slowly, and hence when hit with a weight $\langle x\rangle$ should cost less than the s used in the above calculation. On the other hand, assume that we localize one of the factors $|U|^2$ in $|U|^4 = |U|^2|U|^2$ to relatively large frequency[14]. In that case the key is to use the following simple identity:

$$2is\partial_x[|U|^2](s,.) = (x+2is\partial_x)U(s,.)\bar{U}(s,.) - U(s,.)\overline{(x+2is\partial_x)U(s,.)} \tag{1.13}$$

The operator $C = (x+2is\partial_x)$ is the standard pseudo-conformal operator, and one expects an estimate $\sup_{s\geq 0}||(x+2is\partial_x)U(s,.)||_{L^2_x} \lesssim 1$. Indeed, this turns out to be true (although establishing it requires a couple of tricks, as we don't deal with the free evolution $e^{it\triangle}$ here.) Thus provided we restrict the frequency of $|U|^2(s,.)$ sufficiently far away from 0, we expect to be able to score an extra gain here, and one can play these two considerations against each other to almost obtain the optimal estimate. This very crudely summarizes the strategy for stage **B**.

Stage C: Controlling the root part of $\begin{pmatrix} U \\ \bar{U} \end{pmatrix}$ and the modulation parameters.

We now return to controlling $\lambda_6(t)$ as well as the modulation parameters, which we recall involved estimating expressions such as $\int_T^\infty t\int_t^\infty \langle U^2(s) - \bar{U}^2(s), \phi\rangle ds dt$, as well as similar ones. Clearly even the strong local dispersive estimate isn't good enough for this purpose, and we have to resort to more refined considerations. In case of the displayed expression, this involves identifying another instance of an algebraic cancellation structure, in this case a *symplectic structure*. Again this shall rely on the spectral properties of $\mathcal{H}$.

Stage D: Locating a fixed point.

The a priori estimates suggest running a Banach iteration; unfortunately, the presence of the phase $e^{i(\Psi-\Psi_\infty)(s,y)}$ with $(\Psi-\Psi_\infty)(s,y)$ growing like $s^{\frac{1}{2}+}$ doesn't allow one to deduce good estimates for the differences of iterates. This is the fundamental obstacle to proving Conjecture 1.2. We thus have to resort to an abstract fixed point Theorem (Schauder-Tychonoff Theorem) to prove Theorem 1.3. It is to be hoped that the techniques developed in this paper help to further elucidate the nature of the non-generic blow up solutions.

Acknowledgements: The authors would like to thank Jean Bourgain and Carlos Kenig for helpful discussions and their interest in our work. The first author would also like to express his gratitude to the California Institute of Technology and the University of Chicago for generously hosting him during the Summer 2004 and Summer 2005, respectively.

2. Background material on $\mathcal{H}$.

The material in this section quickly summarizes certain facts established in the last section of [KriSch1], much of which was based on the work of Buslaev-Perelman and Perelman as well as earlier work by the 2nd

---

[14] By this we mean here frequency in the Littlewood-Paley sense. One has to be a bit careful to keep this separate from frequency in the sense of $\mathcal{H}$. The relation of the two will become clear thanks to the explicit distorted Fourier basis explained in the next section; the general heuristic is that a function with small frequency with respect to $\mathcal{H}$ is the sum of a (negligible) Schwartz function plus a function of small frequency in the Littlewood-Paley sense.

author. We refer to [KriSch1] as well as [Per2] for proofs. Consider the operator

$$\mathcal{H} = \begin{pmatrix} \partial_x^2 - 1 + 3\phi_0^4(x,1) & 2\phi_0^4(x,1) \\ -2\phi_0^4(x,1) & -\partial_x^2 + 1 - 3\phi_0^4(x,1) \end{pmatrix}$$

The spectrum consists of $(-\infty,-1]\cup[1,\infty)\cup\{0\}$, with essential spectrum $(-\infty,-1]\cup[1,\infty)$ and discrete spectrum $\{0\}$ of geometric multiplicity 2 and algebraic multiplicity 6. The generalized root space $\mathcal{N}$ is generated by the following vector valued functions: from now on, we adhere to the convention $\phi_0 := \phi_0(.,1)$, see the preceding section.

$$\eta_{1,\text{proper}}(z) := \begin{pmatrix} i\phi_0(z) \\ -i\phi_0(z) \end{pmatrix}, \quad \eta_{2,\text{proper}}(z) := \begin{pmatrix} (z\phi_0'(z) + \phi_0(z)/2) \\ (z\phi_0'(z) + \phi_0(z)/2) \end{pmatrix}$$
$$\eta_{3,\text{proper}}(z) := \begin{pmatrix} \phi_0'(z) \\ \phi_0'(z) \end{pmatrix}, \quad \eta_{4,\text{proper}}(z) := \begin{pmatrix} iz\phi_0(z) \\ -iz\phi_0(z) \end{pmatrix}$$
$$\eta_{5,\text{proper}}(z) := \begin{pmatrix} iz^2\phi_0(z) \\ -iz^2\phi_0(z) \end{pmatrix}, \quad \eta_{6,\text{proper}}(z) := \begin{pmatrix} \rho(z) \\ \rho(z) \end{pmatrix}$$

The first five are in one-one correspondence with internal symmetries('good modes'), while the last is the 'exotic mode', characterized by

$$L_+\rho = z^2\phi_0(z),\ L_+ = -\partial_x^2 + 1 - 5\phi_0^4$$

The root space is generated by $\eta_{1,\text{proper}}$, $\eta_{3,\text{proper}}$.
As for the essential spectrum, its edges ±1 *are no resonances*. This means that there are no non-zero solutions $f_\pm(z) \in L^\infty$ satisfying

$$\mathcal{H} f_\pm = \pm f_\pm$$

This is in marked contrast to the operator $\mathcal{H}_0 := \begin{pmatrix} \partial_x^2 - 1 & 0 \\ 0 & -\partial_x^2 + 1 \end{pmatrix}$, and responsible for much improved local decay estimates.
Identical observations apply to the operator $\mathcal{H}^* = \begin{pmatrix} \partial_x^2 - 1 + 3\phi_0^4(x,1) & -2\phi_0^4(x,1) \\ +2\phi_0^4(x,1) & -\partial_x^2 + 1 - 3\phi_0^4(x,1) \end{pmatrix}$, its generalized root space $\mathcal{N}^*$ being generated by

$$\xi_{1,\text{proper}}(z) := \begin{pmatrix} \phi_0(z) \\ \phi_0(z) \end{pmatrix}, \quad \xi_{2,\text{proper}}(z) := \begin{pmatrix} i(z\phi_0'(z) + \phi_0(z)/2) \\ -i(z\phi_0'(z) + \phi_0(z)/2) \end{pmatrix} \tag{2.1}$$
$$\xi_{3,\text{proper}}(z) := \begin{pmatrix} i\phi_0'(z) \\ -i\phi_0'(z) \end{pmatrix}, \quad \xi_{4,\text{proper}}(z) := \begin{pmatrix} z\phi_0(z) \\ z\phi_0(z) \end{pmatrix} \tag{2.2}$$
$$\xi_{5,\text{proper}}(z) := \begin{pmatrix} z^2\phi_0(z) \\ z^2\phi_0(z) \end{pmatrix}, \quad \xi_{6,\text{proper}}(z) := \begin{pmatrix} i\rho(z) \\ -i\rho(z) \end{pmatrix} \tag{2.3}$$

Then we have the direct sum decomposition

$$L^2(\mathbf{R}) \times L^2(\mathbf{R}) = \mathcal{N} + (\mathcal{N}^*)^\perp$$

This means that every vector valued function $\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(x)$ with $U(.) \in L^2(\mathbf{R})$ can be uniquely represented as

$$\begin{pmatrix} U \\ \bar{U} \end{pmatrix} = \sum_{i=1}^{6} \lambda_i \eta_{i,\text{proper}} + \begin{pmatrix} U \\ \bar{U} \end{pmatrix}_{dis}, \begin{pmatrix} U \\ \bar{U} \end{pmatrix}_{dis} \in (\mathcal{N}^*)^\perp$$

In order to determine the λ_i, one uses the following table of orthogonality relations[15]:

$$\langle \eta_{j,\text{proper}}, \xi_{1,\text{proper}}\rangle = 0,\ 1 \le j \le 5,\ \langle \eta_{6,\text{proper}}, \xi_{1,\text{proper}}\rangle = 2\kappa_2 \tag{2.4}$$

$$\langle \eta_{j,\text{proper}}, \xi_{2,\text{proper}}\rangle = 0,\ j = 1,2,3,4,6, \quad \langle \eta_{5,\text{proper}}, \xi_{2,\text{proper}}\rangle = -4\kappa_2 \tag{2.5}$$

$$\langle \eta_{j,\text{proper}}, \xi_{3,\text{proper}}\rangle = 0,\ j = 1,2,3,5,6, \quad \langle \eta_{4,\text{proper}}, \xi_{3,\text{proper}}\rangle = -\kappa_1 \tag{2.6}$$

$$\langle \eta_{j,\text{proper}}, \xi_{4,\text{proper}}\rangle = 0,\ j = 1,2,4,5,6, \quad \langle \eta_{3,\text{proper}}, \xi_{4,\text{proper}}\rangle = -\kappa_1 \tag{2.7}$$

$$\langle \eta_{j,\text{proper}}, \xi_{5,\text{proper}}\rangle = 0,\ j = 1,3,4,5, \quad \langle \eta_{2,\text{proper}}, \xi_{5,\text{proper}}\rangle = -4\kappa_2, \quad \langle \eta_{6,\text{proper}}, \xi_{5,\text{proper}}\rangle = 2\kappa_3 \tag{2.8}$$

$$\langle \eta_{j,\text{proper}}, \xi_{6,\text{proper}}\rangle = 0,\ j = 2,3,4,6, \quad \langle \eta_{1,\text{proper}}, \xi_{6,\text{proper}}\rangle = 2\kappa_2, \quad \langle \eta_{5,\text{proper}}, \xi_{6,\text{proper}}\rangle = 2\kappa_3, \tag{2.9}$$

where we use the notation

$$\langle \phi_0, \phi_0\rangle = \kappa_1 > 0, \qquad \langle \rho, \phi_0\rangle = \frac{1}{2}\int x^2\phi_0^2(x)\,dx =: \kappa_2 > 0, \qquad \langle x^2\phi_0, \rho\rangle =: \kappa_3,$$

We also write

$$\sum_{i=1}^{6}\lambda_i\eta_{i,\text{proper}} = P_{root}\begin{pmatrix} U \\ \bar{U}\end{pmatrix},\ \begin{pmatrix} U \\ \bar{U}\end{pmatrix}_{dis} = P_s\begin{pmatrix} U \\ \bar{U}\end{pmatrix}$$

We have the following important linear estimates:

Theorem 2.1. *The following estimates hold for vector valued functions $f(.) \in L^1(\mathbf{R})\cap L^2(\mathbf{R})$ and $0 \le \theta \le 1$:*

$$||e^{it\mathcal{H}}P_sf||_{L^2} \lesssim ||f||_{L^2},\ ||e^{it\mathcal{H}}P_sf||_{L^\infty} \lesssim |t|^{-\frac{1}{2}}||f||_{L^1},\ ||\langle x\rangle^{-\theta}e^{it\mathcal{H}}P_sf||_{L^\infty} \lesssim |t|^{-\frac{1}{2}-\theta|}||\langle x\rangle f||_{L^1_x}$$

The first two of these are just as for $e^{it\triangle}$, while the last is not true for the latter and due to the absence of resonances at the edges of the essential spectrum of $\mathcal{H}$.

By analogy to Fourier transformation associated with $\triangle$, there is a *distorted Fourier transform* associated with $\mathcal{H}$:

Theorem 2.2. *There exist Lipschitz continuous vector valued functions $e_\pm(x,\xi)$ with the property*

$$P_s\begin{pmatrix} U \\ \bar{U}\end{pmatrix}(x) = \sum_{\pm}\int_{-\infty}^{\infty} e_\pm(x,\xi)\langle\begin{pmatrix} U \\ \bar{U}\end{pmatrix}, \sigma_3 e_\pm(x,\xi)\rangle d\xi,\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$$

for every rapidly[16] *decaying function $x \to \begin{pmatrix} U \\ \bar{U}\end{pmatrix}(x)$. Moreover, there exist smooth functions $s(\xi)$, $r(\xi)$ satisfying $s(0) = 0$, $r(0) = -1$, as well as suitable numbers $\gamma > 0, \mu > 0$, such that*

$$e_+(x,\xi) = s(\xi)[e^{ix\xi}\begin{pmatrix} 1 \\ 0\end{pmatrix} + O((1+|\xi|)^{-1}e^{-\gamma x})] + O(\xi(1+|\xi|)^{-2}e^{-\mu x}),\ (x,\xi) \in \mathbf{R}_{\ge 0}\times\mathbf{R}_{\ge 0}$$

$$e_+(x,\xi) = [e^{ix\xi} + r(\xi)e^{-ix\xi}]\begin{pmatrix} 1 \\ 0\end{pmatrix} + O(\xi(1+|\xi|)^{-2}e^{\gamma x}),\ (x,\xi) \in \mathbf{R}_{<0}\times\mathbf{R}_{\ge 0}$$

$$e_+(x,\xi) = s(-\xi)[e^{ix\xi}\begin{pmatrix} 1 \\ 0\end{pmatrix} + O((1+|\xi|)^{-1}e^{+\gamma x})] + O(\xi(1+|\xi|)^{-2}e^{+\mu x}),\ (x,\xi) \in \mathbf{R}_{<0}\times\mathbf{R}_{<0}$$

$$e_+(x,\xi) = [e^{ix\xi} + r(-\xi)e^{-ix\xi}]\begin{pmatrix} 1 \\ 0\end{pmatrix} + O(\xi(1+|\xi|)^{-2}e^{-\gamma x}),\ (x,\xi) \in \mathbf{R}_{\ge 0}\times\mathbf{R}_{<0}$$

Also, we have the relation

$$e_-(x,\xi) = \sigma_1 e_+(x,\xi),\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$$

In analogy to the usual Fourier transform, there is a *distorted Plancherel's Theorem*:

[15]We use the convention $\langle\begin{pmatrix} U_1 \\ U_2\end{pmatrix}, \begin{pmatrix} V_1 \\ V_2\end{pmatrix}\rangle = \langle U_1, V_1\rangle + \langle U_2, V_2\rangle$.

[16]We are being overly restrictive in the formulation here; all facts about the distorted Fourier transform apply in the same degree of generality as for the ordinary Fourier transform.

Theorem 2.3. *Let $\phi, \psi \in \mathcal{S}(\mathbf{R})$ be vector valued functions. Then we have*

$$\langle P_s\phi, \psi\rangle = \sum_{\pm} \frac{1}{2\pi}\int_{-\infty}^{\infty} \langle \phi, \sigma_3 e_\pm(.,\xi)\rangle \overline{\langle \psi, e_\pm(.,\xi)\rangle} d\xi$$

We shall use the notation

$$\mathcal{F}_\pm(\phi)(\xi) := \langle \phi, \sigma_3 e_\pm(.,\xi)\rangle,\ \tilde{\mathcal{F}}_\pm(\phi)(\xi) := \langle \phi, e_\pm(x,\xi)\rangle$$

When working with $e_\pm(x,\xi)$, we shall for example write

$$e_+(x,\xi) = s(\xi)e^{ix\xi}\underline{e} + \phi(x,\xi),\ (x,\xi) \in \mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0},$$

and similarly for the other values of (x,ξ). The functions $\phi(x,\xi)$, which are Schwartz with respect to x, are understood to vary accordingly, but all vanish uniformly in x at $\xi = 0$ and decay like $|\xi|^{-1}$ for $|\xi| \to \infty$ uniformly in x.

3. Setting up the equations

3.1. **Algebraic manipulations I; analysis of the modulation parameters.** We now flesh out the discussion of the first section. In this section as well as the next, we shall use formal algebraic manipulations to derive the equations which will serve to define the iterative step. We shall also mention the required estimates. In the final sections of the paper, we shall then show that the iterative step indeed makes sense when performed on suitable function spaces. Thus consider now a solution $\psi(t,x) = W(t,x)+R(t,x)$ of (1.1), where

$$W(t,x) = e^{i(\theta(t,x))}e^{-i\frac{\beta}{4}(t)\lambda^2(x-\mu(t))^2}\sqrt{\lambda(t)}\phi_0(\lambda(t)(x-\mu(t)))$$

Use the notation $\Psi(t,x) = \theta(t,x) - \frac{\beta}{4}\lambda^2(x-\mu)^2$, $z = \lambda(x-\mu)$. Also, write $\tilde{\theta}(t,z) := \theta(t,x)$. An elementary calculation then shows that we have

$$\begin{aligned}
&i\partial_t W + \partial_x^2 W + |W|^4 W\\
&= (\lambda_t\lambda^{-1} - \beta\lambda^2)[ie^{i\Psi}z\lambda^{\frac{1}{2}}\phi_0'(z) + \frac{\beta}{2}z^2 W + \frac{i}{2}W - \tilde{\theta}_z z W]\\
&+ \frac{z^2}{4}(\beta_t + \lambda^2\beta^2)W + i\lambda^2\tilde{\theta}_{zz}W + (\lambda^2 - \tilde{\theta}_t + \lambda^2\tilde{\theta}_z^2 - \beta\lambda^2 z\tilde{\theta}_z)W\\
&+ (\mu_t - 2\lambda\tilde{\theta}_z)[\lambda\tilde{\theta}_z - i\lambda^{\frac{3}{2}}e^{i\Psi}\phi_0'(z) - \frac{\beta}{2}\lambda z W]
\end{aligned}$$

Introduce the following notation:

$$\eta_1 := \begin{pmatrix} ie^{i\Psi}\sqrt{\lambda}\phi_0(z)\\ -ie^{-i\Psi}\sqrt{\lambda}\phi_0(z)\end{pmatrix}, \quad \eta_2 := \begin{pmatrix} e^{i\Psi}(z\sqrt{\lambda}\phi_0'(z) + \sqrt{\lambda}\phi_0(z)/2)\\ e^{-i\Psi}(z\sqrt{\lambda}\phi_0'(z) + \sqrt{\lambda}\phi_0(z)/2)\end{pmatrix}$$
$$\eta_3 := \begin{pmatrix} e^{i\Psi}\sqrt{\lambda}\phi_0'(z)\\ e^{-i\Psi}\sqrt{\lambda}\phi_0'(z)\end{pmatrix}, \quad \eta_4 := \begin{pmatrix} ie^{i\Psi}z\sqrt{\lambda}\phi_0(z)\\ -ie^{-i\Psi}z\sqrt{\lambda}\phi_0(z)\end{pmatrix}$$
$$\eta_5 := \begin{pmatrix} ie^{i\Psi}z^2\sqrt{\lambda}\phi_0(z)\\ -ie^{-i\Psi}z^2\sqrt{\lambda}\phi_0(z)\end{pmatrix}, \quad \eta_6 := \begin{pmatrix} e^{i\Psi}\sqrt{\lambda}\rho(z)\\ e^{-i\Psi}\sqrt{\lambda}\rho(z)\end{pmatrix}$$

Now impose the relation $\tilde{\theta}_{zz} = 0$, whence $\tilde{\theta} = \gamma(t) + \omega(t)z$. The function ρ here is defined via

$$L_- := -\partial_{xx} + 1 - \phi_0^4, \qquad L_+ := -\partial_{xx} + 1 - 5\phi_0^4$$
$$L_-\phi_0 = 0, \quad L_+(\frac{1}{2}\phi_0 + x\phi_0') = -2\phi_0$$
$$L_-(x^2\phi_0) = -4(\frac{1}{2}\phi_0 + x\phi_0'), \quad L_+\rho = x^2\phi_0$$

Then the vector-function $\mathcal{W}(t,x) := \binom{W(t,x)}{\bar{W}(t,x)}$ satisfies

$$
\begin{aligned}
&i\partial_t \mathcal{W} + \begin{bmatrix} \partial_{xx} & 0 \\ 0 & -\partial_{xx} \end{bmatrix} \mathcal{W} + \begin{pmatrix} |W|^4 W \\ -|W|^4 \bar{W} \end{pmatrix} \\
&= i(\dot{\lambda}\lambda^{-1} - \beta\lambda^2)(\eta_2 - \beta\eta_5/2 + \omega\eta_4) - \frac{i}{4}(\dot{\beta} + \lambda^2\beta^2)\eta_5 - i(\lambda^2 - \dot{\gamma} + \lambda^2\omega^2)\eta_1 \\
&\quad + i(\dot{\omega} + \beta\lambda^2\omega)\eta_4 + \lambda(\dot{\mu} - 2\lambda\omega)(-i\omega\eta_1 - i\eta_3 + i\beta\eta_4/2)
\end{aligned}
$$

Write $\begin{pmatrix} \psi(t,x) \\ \overline{\psi(t,x)} \end{pmatrix} = \mathcal{W}(t,x) + Z(t,x)$ where $Z(t,x) = \begin{pmatrix} R(t,x) \\ \overline{R(t,x)} \end{pmatrix}$. We deduce the following equation for Z:

$$
i\partial_t Z + \mathcal{H}(t)Z = -i(\dot{\lambda}\lambda^{-1} - \beta\lambda^2)(\eta_2 - \beta\eta_5/2 + \omega\eta_4) + \frac{i}{4}(\dot{\beta} + \lambda^2\beta^2)\eta_5 + i(\lambda^2 - \dot{\gamma} + \lambda^2\omega^2)\eta_1 \tag{3.1}
$$

$$
- i(\dot{\omega} + \beta\lambda^2\omega)\eta_4 - \lambda(\dot{\mu} - 2\lambda\omega)(-i\omega\eta_1 - i\eta_3 + i\beta\eta_4/2) + N(Z) \tag{3.2}
$$

$$
\mathcal{H}(t) := \begin{bmatrix} \partial_{xx} + 3|W|^4 & 2|W|^2 W^2 \\ -2|W|^2 \bar{W}^2 & -\partial_{xx} - 3|W|^4 \end{bmatrix} \tag{3.3}
$$

$$
N(Z) := \begin{pmatrix} -|R+W|^4(R+W) + |W|^4 W + 3|W|^4 R + 2|W|^2 W^2 \bar{R} \\ |R+W|^4(\bar{R}+\bar{W}) - |W|^4\bar{W} - 3|W|^4\bar{R} - 2|W|^2\bar{W}^2 R \end{pmatrix} \tag{3.4}
$$

$$
= \begin{pmatrix} -3R^2|W|^2\bar{W} - 6|R|^2|W|^2 W - W^3\bar{R}^2 + O(|R|^3|W|^2 + |R|^5) \\ 3\bar{R}^2|W|^2 W + 6|R|^2|W|^2\bar{W} + \bar{W}^3 R^2 + O(|R|^3|W|^2 + |R|^5) \end{pmatrix} \tag{3.5}
$$

In order to formulate the modulation equations, it will be convenient to introduce the following family of auxiliary functions, which are in some sense dual to the η_i:

$$
\begin{aligned}
\xi_1 &:= \begin{pmatrix} e^{i\Psi}\sqrt{\lambda}\phi_0(z) \\ e^{-i\Psi}\sqrt{\lambda}\phi_0(z) \end{pmatrix}, \quad \xi_2 := \begin{pmatrix} ie^{i\Psi}(z\sqrt{\lambda}\phi_0'(z) + \sqrt{\lambda}\phi_0(z)/2) \\ -ie^{-i\Psi}(z\sqrt{\lambda}\phi_0'(z) + \sqrt{\lambda}\phi_0(z)/2) \end{pmatrix} \\
\xi_3 &:= \begin{pmatrix} ie^{i\Psi}\sqrt{\lambda}\phi_0'(z) \\ -ie^{-i\Psi}\sqrt{\lambda}\phi_0'(z) \end{pmatrix}, \quad \xi_4 := \begin{pmatrix} e^{i\Psi}z\sqrt{\lambda}\phi_0(z) \\ e^{-i\Psi}z\sqrt{\lambda}\phi_0(z) \end{pmatrix} \\
\xi_5 &:= \begin{pmatrix} e^{i\Psi}z^2\sqrt{\lambda}\phi_0(z) \\ e^{-i\Psi}z^2\sqrt{\lambda}\phi_0(z) \end{pmatrix}, \quad \xi_6 := \begin{pmatrix} ie^{i\Psi}\sqrt{\lambda}\rho(z) \\ -ie^{-i\Psi}\sqrt{\lambda}\rho(z) \end{pmatrix}
\end{aligned}
$$

Then, analogously to (2.4), we have the following (using the same notation)

$$
\langle \eta_j, \xi_1 \rangle = 0,\ 1 \le j \le 5,\ \langle \eta_6, \xi_1 \rangle = 2\kappa_2 \tag{3.6}
$$

$$
\langle \eta_j, \xi_2 \rangle = 0,\ j = 1,2,3,4,6, \quad \langle \eta_5, \xi_2 \rangle = -4\kappa_2 \tag{3.7}
$$

$$
\langle \eta_j, \xi_3 \rangle = 0,\ j = 1,2,3,5,6, \quad \langle \eta_4, \xi_3 \rangle = -\kappa_1 \tag{3.8}
$$

$$
\langle \eta_j, \xi_4 \rangle = 0,\ j = 1,2,4,5,6, \quad \langle \eta_3, \xi_4 \rangle = -\kappa_1 \tag{3.9}
$$

$$
\langle \eta_j, \xi_5 \rangle = 0,\ j = 1,3,4,5, \quad \langle \eta_2, \xi_5 \rangle = -4\kappa_2, \quad \langle \eta_6, \xi_5 \rangle = 2\kappa_3 \tag{3.10}
$$

$$
\langle \eta_j, \xi_6 \rangle = 0,\ j = 2,3,4,6, \quad \langle \eta_1, \xi_6 \rangle = 2\kappa_2, \quad \langle \eta_5, \xi_6 \rangle = 2\kappa_3 \tag{3.11}
$$

Recall from the discussion in the first section that we impose the orthogonality relations $\langle Z, \xi_i \rangle = 0$, $i = 2, \ldots, 6$. This allows us to control the 'good component' of the root part of the radiation. Using Leibnitz' rule we get

$$
\langle i\partial_t Z + \mathcal{H}(t)Z, \xi_j(t) \rangle = \langle Z, (i\partial_t + \mathcal{H}(t)^*)\xi_j(t) \rangle =: \langle Z, \mathcal{L}^*\xi_j \rangle
$$

Explicitly, using (3.1), these read as follows:

$$
\begin{aligned}
-2i\beta(\dot\lambda\lambda^{-1}-\beta\lambda^2)\kappa_2 - i(\dot\beta+\lambda^2\beta^2)\kappa_2 &= \langle Z,\mathcal{L}^*\xi_2\rangle - \langle N(Z),\xi_2\rangle\\
\frac{i}{2}\lambda\beta\kappa_1(\dot\mu-2\lambda\omega) + i\omega\kappa_1(\dot\lambda\lambda^{-1}-\beta\lambda^2) + i\kappa_1(\dot\omega+\beta\lambda^2\omega) &= \langle Z,\mathcal{L}^*\xi_3\rangle - \langle N(Z),\xi_3\rangle\\
-i\lambda\kappa_1(\dot\mu-2\lambda\omega) &= \langle Z,\mathcal{L}^*\xi_4\rangle - \langle N(Z),\xi_4\rangle\\
4i\kappa_2(\dot\lambda\lambda^{-1}-\beta\lambda^2) &= \langle Z,\mathcal{L}^*\xi_5\rangle - \langle N(Z),\xi_5\rangle\\
2i\kappa_2(\lambda^2-\dot\gamma+\lambda^2\omega^2) + 2i\lambda\omega\kappa_2(\dot\mu-2\lambda\omega) + i\beta\kappa_3(\dot\lambda\lambda^{-1}-\beta\lambda^2) + \frac{i}{2}\kappa_3(\dot\beta+\lambda^2\beta^2)&\\
= \langle Z,\mathcal{L}^*\xi_6\rangle - \langle N(Z),\xi_6\rangle&
\end{aligned}
$$

Before proceeding, let's carry out a consistency check: we know that the case $Z=0$ corresponds to a transformed standing wave. In this case, the above relations simplify to

$$
\begin{aligned}
&\dot\lambda\lambda^{-1}-\beta\lambda^2=0,\quad \dot\beta+\lambda^2\beta^2=0,\quad \frac{d}{dt}(\lambda\beta)=0,\ \beta=-b\lambda^{-1}\\
&\dot\lambda=-b\lambda^2,\quad \lambda(t)=(a+bt)^{-1},\ \beta(t)=-b(a+bt),\\
&\dot\mu-2\lambda\omega=0,\quad \dot\omega+\beta\lambda^2\omega=0,\quad \dot\omega-b\lambda\omega=0\\
&\omega(t)=v(a+bt),\quad \dot\mu=2\lambda\omega=2v,\quad \mu(t)=2tv+\mu_0\\
&\lambda^2-\dot\gamma+\lambda^2\omega^2=0,\ \dot\gamma=\lambda^2+v^2=-\frac{1}{b}\dot\lambda+v^2\\
&\gamma(t)=-\frac{\lambda}{b}+tv^2+\gamma_0=-\frac{1}{b(a+bt)}+v^2t+\gamma_0
\end{aligned}
$$

So the exact solution looks as follows (with $\gamma=\gamma_0-v\mu_0$):

$$
\begin{aligned}
\theta(t,z)&=-\frac{1}{b(a+bt)}-v^2t+\gamma+vx\\
\lambda(t)&=(a+bt)^{-1}\\
\beta(t)&=-b(a+bt)\\
\mu(t)&=2tv+\mu_0
\end{aligned}
$$

and the transformed standing wave W is

$$
\begin{aligned}
W(t,x)=\exp\Big(i\Big[-\frac{1}{b(a+bt)}-v^2t+\gamma+vx+\frac{b}{4(a+bt)}(x-2tv-\mu_0)^2\Big]\Big)\cdot&\\
(a+bt)^{-\frac12}\phi_0\big((a+bt)^{-1}(x-2tv-\mu_0)\big).& \qquad (3.12)
\end{aligned}
$$

Now recall the pseudo-conformal transformation

$$\mathcal{C}_M:\quad \psi(t,x)\to(a+bt)^{-\frac12}\exp\Big(i\frac{bx^2}{4(a+bt)}\Big)\psi\Big(\frac{c+dt}{a+bt},\frac{x}{a+bt}\Big)$$

where $M=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{R})$. Starting from the standing wave $e^{it}\phi_0(x)$, apply the pseudo-conformal transformation with matrix $\begin{pmatrix} a & b\\ -b^{-1} & 0\end{pmatrix}$:

$$\exp\Big(-i\frac{1}{b(a+bt)}+i\frac{b}{4(a+bt)}x^2\Big)(a+bt)^{-\frac12}\phi_0((a+bt)^{-1}x)$$

and then the Galilei transform

$$\mathfrak{g}_{\gamma,v,\mu_0}(t)=e^{i(\gamma+vx-tv^2)}e^{-i(2tv+\mu_0)p}.$$

This leads to the exact same expression as in (3.12).

We now intend to translate the above equations from the (t,x) coordinates to a new coordinate system (s,y), in which we 'de-singularize' the equations. The blow-up time t_* shall be transformed into $s=+\infty$, and the t-interval $(-\infty,t_*]$ shall correspond to $(-c,\infty]$ for suitable $c>0$. Thus make the ansatz[17]

$$\begin{pmatrix} U \\ \bar{U} \end{pmatrix} := \mathcal{M}\mathcal{T}_\infty \begin{pmatrix} R \\ \bar{R} \end{pmatrix},$$

where

$$\mathcal{T}_\infty = e^{-i(v_\infty^2 s+\gamma_\infty+v_\infty y)} e^{i(2v_\infty s+y_\infty)p} \begin{pmatrix} a_\infty & b_\infty \\ 0 & a_\infty^{-1} \end{pmatrix}, \mathcal{M}=\mathcal{M}(s)=\begin{pmatrix} e^{-is} & 0 \\ 0 & e^{is} \end{pmatrix}, p=-i\frac{d}{dy}$$

Then we have the succinct identity

$$(\mathcal{T}_\infty F)(s,y) = e^{-i\tilde{\Psi}_\infty(s,y)} \lambda_\infty^{-\frac{1}{2}}(s) F\Big(\int_0^s \lambda_\infty^{-2}(\sigma)d\sigma, \lambda_\infty^{-1}(s)y+\mu_\infty(s)\Big),$$

where

$$\tilde{\Psi}_\infty(s,y) = v_\infty^2 s+\gamma_\infty+v_\infty y - \frac{b_\infty(y+2v_\infty s+y_\infty)^2}{4(a_\infty+b_\infty s)},\ \mu_\infty(s)=\frac{2v_\infty s+y_\infty}{a_\infty+b_\infty s},\ \lambda_\infty = a_\infty+b_\infty s \tag{3.13}$$

Thus we have

$$e^{-is}(\mathcal{T}_\infty W)(s,y) = e^{-i(s+\tilde{\Psi}_\infty)+i\Psi(t(s),\mu_\infty+\lambda_\infty^{-1}(s)y)} \lambda_\infty^{-\frac{1}{2}}(s)\lambda(t(s))^{\frac{1}{2}}\phi_0(\lambda(t(s))(\mu_\infty(s)-\mu(t(s))+\lambda_\infty^{-1}(s)y)),$$

where we put $t(s) := \int_0^s \lambda_\infty^{-2}(\sigma)d\sigma = \frac{a_\infty^{-1}s}{a_\infty+b_\infty s}$. We shall now impose the asymptotic conditions $\nu(s) := \frac{\lambda(s)}{\lambda_\infty(s)} \to 1$, $\lambda(t(s))(\mu_\infty(s)-\mu(t(s))) \to 0$ as $s\to+\infty$. Unfortunately, it appears that no such requirement can be applied to $\Psi(t(s),\mu_\infty+\lambda_\infty^{-1}(s)y)-s-\tilde{\Psi}_\infty(s,y)$, as will follow from the ensuing discussion.

Now introduce the Schwartz functions $\tilde{\eta}_i := \mathcal{M}\mathcal{T}_\infty\eta_i$, $\tilde{\xi}_i := \mathcal{M}\mathcal{T}_\infty\xi_i$. Then we can deduce the following equation for $\begin{pmatrix} U \\ \bar{U} \end{pmatrix}$:

$$i\partial_s \begin{pmatrix} U \\ \bar{U} \end{pmatrix} + \mathcal{H}(s)\begin{pmatrix} U \\ \bar{U} \end{pmatrix} = -i(\dot{\lambda}\lambda^{-1}-\beta\nu^2)(\tilde{\eta}_2-\beta\tilde{\eta}_5/2+\omega\tilde{\eta}_4) \tag{3.14}$$

$$+\frac{i}{4}(\dot{\beta}+\beta^2\nu^2)\tilde{\eta}_5 + i(\nu^2-\dot{\gamma}+\nu^2\omega^2)\tilde{\eta}_1 \tag{3.15}$$

$$-i(\dot{\omega}+\beta\omega\nu^2)\tilde{\eta}_4 - i\nu(\dot{\mu}\lambda_\infty-2\nu\omega)(-\omega\tilde{\eta}_1-\tilde{\eta}_3+\beta\tilde{\eta}_4/2)+N(U,\pi), \tag{3.16}$$

In this equation $\dot{\lambda}=\partial_s[\lambda(t(s))]$. We use the abbreviations[18] (with $\begin{pmatrix} U \\ \bar{U} \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$)

$$N(U,\pi) := \begin{pmatrix} -3U_1^2\tilde{\phi}_0^3 e^{i(\Psi_\infty-\Psi)}\nu^{\frac{3}{2}} - 6|U_1|^2 e^{i(\Psi-\Psi_\infty)}\nu^{\frac{3}{2}}\tilde{\phi}_0^3 - U_2^2 e^{3i(\Psi-\Psi_\infty)}\nu^{\frac{3}{2}}\tilde{\phi}_0^3 + O(|U|^3+|U|^5) \\ 3U_1^2\tilde{\phi}_0^3 e^{-i(\Psi_\infty-\Psi)}\nu^{\frac{3}{2}} + 6|U_1|^2 e^{-i(\Psi-\Psi_\infty)}\nu^{\frac{3}{2}}\tilde{\phi}_0^3 + U_2^2 e^{-3i(\Psi-\Psi_\infty)}\nu^{\frac{3}{2}}\tilde{\phi}_0^3 + O(|U|^3+|U|^5) \end{pmatrix}$$

where $\tilde{\phi}_0(y) = \phi_0(\lambda(\mu_\infty-\mu+\lambda_\infty^{-1}y))$.

$$\mathcal{H}(s) := \begin{pmatrix} \partial_y^2-1+3\nu^2(s)\tilde{\phi}_0^4 & 2\nu^2\tilde{\phi}_0^4 e^{2i(\Psi-\Psi_\infty)} \\ -2\nu^2\tilde{\phi}_0^4 e^{-2i(\Psi-\Psi_\infty)} & -\partial_y^2+1-3\nu^2(s)\tilde{\phi}_0^4 \end{pmatrix}$$

The orthogonality conditions $\langle Z,\xi_i\rangle = 0$, $i=2,\dots,6$, translate to $\langle \begin{pmatrix} U \\ \bar{U} \end{pmatrix}, \tilde{\xi}_i\rangle = 0$, $i=2,\dots,6$. If one differentiates this relation with respect to s and uses the Leibnitz rule as well as (3.14), this leads to the

[17] We use the notation $U(s,y)$, $U(s)$, U *all for the same function of two variables* in order to streamline the notation in some places.

[18] Also, recall that $\Psi_\infty(s,y) = \tilde{\Psi}_\infty(s,y)+s$.

following system of 'ODE's', the modulation equations:

$$-2\kappa_2 i\beta(\dot\lambda\lambda^{-1}-\beta\nu^2)-i\kappa_2(\dot\beta+\beta^2\nu^2)=-\langle N,\tilde\xi_2\rangle+\langle U,(i\partial_s+\mathcal{H}(s)^*)\tilde\xi_2\rangle \tag{3.17}$$

$$i\omega\kappa_1(\dot\lambda\lambda^{-1}-\beta\nu^2)+i\kappa_1(\dot\omega+\beta\omega\nu^2)+\frac{i}{2}\beta\nu(\dot\mu\lambda_\infty-2\omega\nu)\kappa_1=-\langle N,\tilde\xi_3\rangle+\langle U,(i\partial_s+\mathcal{H}(s)^*)\tilde\xi_3\rangle \tag{3.18}$$

$$-i\kappa_1\nu(\dot\mu\lambda_\infty-2\omega\nu)=-\langle N,\tilde\xi_4\rangle+\langle U,(i\partial_s+\mathcal{H}(s)^*)\tilde\xi_4\rangle \tag{3.19}$$

$$4i\kappa_2(\dot\lambda\lambda^{-1}-\beta\nu^2)=-\langle N,\tilde\xi_5\rangle+\langle U,(i\partial_s+\mathcal{H}(s)^*)\tilde\xi_5\rangle \tag{3.20}$$

$$\begin{aligned}2i\kappa_2(\nu^2-\dot\gamma+\nu^2\omega^2)+2i\omega\nu\kappa_2(\dot\mu\lambda_\infty-2\nu\omega)+\frac{i}{2}\kappa_3(\dot\beta+\beta^2\nu^2)+i\beta\kappa_3(\dot\lambda\lambda^{-1}-\beta\nu^2)\\=-\langle N,\tilde\xi_6\rangle+\langle U,(i\partial_s+\mathcal{H}(s)^*)\tilde\xi_6\rangle,\end{aligned} \tag{3.21}$$

Of course, for all this to make sense we need to specify the 'parameters at infinity' $\{a_\infty,b_\infty,v_\infty,y_\infty,\gamma_\infty\}$. We shall soon see that their value is forced by the asymptotic conditions on the modulation parameters.

In accordance with the statement of Theorem 1.3, we now fix the values of $\lambda(s)$,$\beta(s)$,$\mu(s)$, $\omega(s)$, $\gamma(s)$ at time $s=0$, where we require $\lambda(0)\sim 1$, $\beta(0)\sim 1$. Instead of working with these parameters, though, we shall work with $\nu(s)=\frac{\lambda(s)}{\lambda_\infty(s)}$, $\beta(s)\nu(s)-\frac{b_\infty}{\lambda_\infty(s)}$, $\mu(s)$, $\omega(s)$, $\gamma(s)$. Start with the fourth modulation equation. Formulate this as follows:

$$\dot\nu\nu^{-1}-\beta\nu^2=(4i\kappa_2)^{-1}[-\langle N,\tilde\xi_5\rangle+\langle U,(i\partial_s+\mathcal{H}(s)^*)\tilde\xi_5\rangle]-b_\infty\lambda_\infty^{-1},$$

From the fourth and 2nd equation, we get

$$\dot\beta+\beta^2\nu^2=-(i\kappa_2)^{-1}E_2-\frac{\beta}{2i\kappa_2}E_5,$$

where we use the notation $E_j:=-\langle N,\tilde\xi_j\rangle+\langle U,(i\partial_s+\mathcal{H}^*(s))\tilde\xi_j\rangle$. Noting the simple identity $(b_\infty\lambda_\infty^{-1})_s+(b_\infty\lambda_\infty^{-1})^2=0$, we get

$$\dot\nu-b_\infty\lambda_\infty^{-1}(\nu-1)=\nu(4i\kappa_2)^{-1}E_5+\beta\nu(\nu-1)^2+(2\nu-1)(\beta\nu-b_\infty\lambda_\infty^{-1})$$

$$\frac{d}{ds}(\beta\nu-b_\infty\lambda_\infty^{-1})+(\beta\nu-b_\infty\lambda_\infty^{-1})b_\infty\lambda_\infty^{-1}=-\nu(i\kappa_2)^{-1}E_2+[-\beta\frac{\nu}{2i\kappa_2}+\nu\beta(4i\kappa_2)^{-1}]E_5$$

We can further reformulate these equations as follows:

$$\frac{d}{ds}[(\nu-1)\lambda_\infty^{-1}](s)=\lambda_\infty^{-1}[\nu(4i\kappa_2)^{-1}E_5+\beta\nu(\nu-1)^2+(2\nu-1)(\beta\nu-b_\infty\lambda_\infty^{-1})]$$

$$\frac{d}{ds}([\beta\nu-b_\infty\lambda_\infty^{-1}]\lambda_\infty)=\lambda_\infty[-\nu(i\kappa_2)^{-1}E_2-\frac{\beta\nu}{4i\kappa_2}E_5](s)$$

The condition that $\nu(s)\to 1$ as $s\to+\infty$, as well as the condition $\beta\nu(s)-b_\infty\lambda_\infty^{-1}(s)\to 0$ imply the following identities:

$$0=(\beta\nu-b_\infty\lambda_\infty^{-1})\lambda_\infty(0)+\int_0^\infty\lambda_\infty(s)[-\nu(i\kappa_2)^{-1}E_2-\frac{\beta\nu}{4i\kappa_2}E_5](s)ds$$

$$0=(\nu-1)\lambda_\infty^{-1}(0)+\int_0^\infty\lambda_\infty(s)^{-1}[\nu(4i\kappa_2)^{-1}E_5+\beta\nu(\nu-1)^2+(2\nu-1)(\beta\nu-b_\infty\lambda_\infty^{-1})](s)ds$$

whence

$$0=(\beta(0)\lambda(0)-b_\infty)+\int_0^\infty\lambda_\infty(s)[-\nu(i\kappa_2)^{-1}E_2+\frac{3\beta\nu}{4i\kappa_2}E_5](s)ds \tag{3.22}$$

$$0=\lambda(0)-a_\infty+a_\infty^2\int_0^\infty\lambda_\infty(s)^{-1}[\nu(4i\kappa_2)^{-1}E_5+\beta\nu(\nu-1)^2+(2\nu-1)(\beta\nu-b_\infty\lambda_\infty^{-1})](s)ds \tag{3.23}$$

Assuming the integral expressions known, this allows for solving for the coefficients a_∞,b_∞, using the Implicit function Theorem. Moreover, we get the formulae

$$\nu(s)-1=-\lambda_\infty(s)\int_s^\infty\lambda_\infty(\sigma)^{-1}[\nu(4i\kappa_2)^{-1}E_5+\beta\nu(\nu-1)^2+(2\nu-1)(\beta\nu-b_\infty\lambda_\infty^{-1})](\sigma)d\sigma \tag{3.24}$$

$$(\beta\nu - b_\infty\lambda_\infty^{-1})(s) = -\lambda_\infty(s)^{-1}\int_s^\infty \lambda_\infty(\sigma)[-\nu(i\kappa_2)^{-1}E_2 - \frac{\beta\nu}{4i\kappa_2}E_5](\sigma)d\sigma \tag{3.25}$$

Next, from the 2nd, 3rd and 4th modulation equation we gather

$$\dot{\omega} + \beta\nu^2\omega = (i\kappa_1)^{-1}E_3 - \omega(4i\kappa_2)^{-1}E_5 + \frac{\beta}{2i\kappa_1}E_4$$

Introduce the quantity

$$B(s) = \exp(\int_0^s [\beta\nu^2 + \frac{1}{4i\kappa_2}E_5](\sigma)d\sigma)$$

We can then write

$$\omega(s) = B(s)^{-1}\omega(0) + \int_0^s \frac{B(\sigma)}{B(s)}[(i\kappa_1)^{-1}E_3 + \frac{\beta}{2i\kappa_1}E_4](\sigma)d\sigma \tag{3.26}$$

Decompose

$$\beta\nu^2 = (\beta\nu - b_\infty\lambda_\infty^{-1})\nu + (\nu-1)b_\infty\lambda_\infty^{-1} + b_\infty\lambda_\infty^{-1}$$

The stipulations $\lim_{s\to+\infty}\nu(s) = 1$, $\lim_{s\to+\infty}\beta\nu - b_\infty\lambda_\infty^{-1} = 0$ then yield[19]

$$B^{-1}(s) = c\lambda_\infty^{-1}(s) + o(\frac{1}{s}) \tag{3.27}$$

We then reformulate (3.26) as follows:

$$\begin{aligned}\omega(s) = c\lambda_\infty(s)^{-1}[\omega(0) + \int_0^\infty B(\sigma)[(i\kappa_1)^{-1}E_3 + \frac{\beta}{2i\kappa_1}E_4](\sigma)d\sigma]\\ - c\lambda_\infty(s)^{-1}\int_s^\infty B(\sigma)[(i\kappa_1)^{-1}E_3 + \frac{\beta}{2i\kappa_1}E_4](\sigma)d\sigma] + o(s^{-1}),\end{aligned} \tag{3.28}$$

from which we obtain for suitable c_∞ the asymptotic relation $\omega(s) = c_\infty\lambda_\infty^{-1} + o(s^{-1})$, provided we can control all the integrals. From the 3rd modulation equation we obtain

$$\mu(s) = \mu(0) + \int_0^s \lambda_\infty(\sigma)^{-1}[2\omega\nu - (i\kappa_1\nu(\sigma))^{-1}E_4(\sigma)]d\sigma \tag{3.29}$$

If we feed in the relation (3.27), we infer the existence of parameters v_∞, y_∞ with the property

$$\mu(s) = \frac{2v_\infty s + y_\infty}{a_\infty + b_\infty s} + o(s^{-1}) \tag{3.30}$$

Finally, the 5th modulation equation gives

$$\gamma(s) = \gamma(0) + \int_0^s [\nu^2(\sigma) - (2i\kappa_2)^{-1}E_6(\sigma) + \nu^2(\sigma)\omega^2(\sigma) - \frac{1}{i\kappa_1}\omega(\sigma)E_4(\sigma) - \frac{1}{2\kappa_2}(i\kappa_2)^{-1}E_2(\sigma)]d\sigma \tag{3.31}$$

Last but not least, we choose γ_∞ such that $(\Psi - \Psi_\infty)_1(0) = 0$, where $\Psi_\infty(s,y) = \tilde{\Psi}_\infty(s,y) + s$, and we define $(\Psi - \Psi_\infty)_1(s)$ to be that part of $\Psi - \Psi_\infty$ which only depends on s, see the ensuing subsection.

We now state the **precise estimates for the modulation parameters**: first, choose small positive numbers δ_i, $i = 1,2,3$, and $\delta > 0$ with the property $\delta << \delta_2 << \delta_3 << \delta_1$. These shall be fixed throughout. The number δ will control the size[20] of radiation part as well as modulation parameters, while the parameters δ_i, measure parameters in certain norms. Then we need for a sufficiently large[21] $N = N(\delta_2, \delta_3, \delta_1)$ and very

[19] We shall soon specify the precise decay rates.

[20] With respect to suitable norms.

[21] We shall need $N << N_1(\delta_2,\delta_1)$ and $N > N_2(\delta_3)$. The parameter δ_2 will appear in the estimates (3.48), where we specify $N_1(\delta_2,\delta_1) \sim |\log(\frac{\delta_1}{\delta_2})|$. The bound $N > N_2(\delta_3)$ is needed in bootstrapping the strong local dispersive estimate.

large $M > M(\delta_2)$ held fixed throughout

$$
\begin{aligned}
&|\nu(s)-1| \lesssim \delta^2 \langle s\rangle^{-\frac{1}{2}+\delta_1}, \sup_{1\leq i\leq[\frac{N}{3}]} ||\langle s\rangle^{\frac{3}{2}-2\delta_1}\frac{d^i}{ds^i}\nu(s)||_{L^M} \lesssim \delta^2,\ |\beta(s)\nu(s)-b_\infty\lambda_\infty^{-1}(s)| \lesssim \langle s\rangle^{-\frac{3}{2}+\delta_1}\delta^2,\\
&\sup_{1\leq i\leq[\frac{N}{3}]} ||\langle s\rangle^{2-2\delta_1}\frac{d^i}{ds^i}[\beta(s)\nu(s)-b_\infty\lambda_\infty^{-1}(s)]||_{L^M} \lesssim \delta^2,\ |\omega(s)-c_\infty\lambda_\infty^{-1}(s)| \lesssim \delta^2\langle s\rangle^{-\frac{3}{2}+\delta_1},\\
&\sup_{1\leq i\leq[\frac{N}{3}]} ||\langle s\rangle^{2-2\delta_1}\frac{d^i}{ds^i}[\omega(s)-c_\infty\lambda_\infty^{-1}(s)]||_{L^M} \lesssim \delta^2, |\partial_s(\gamma(s)-s)-c_\infty^2\lambda_\infty^{-2}(s)| \lesssim \delta^2\langle s\rangle^{-\frac{1}{2}+\delta_1},\\
&\sup_{1\leq i\leq[\frac{N}{3}]} ||\langle s\rangle^{\frac{3}{2}-2\delta_1}\frac{d^i}{ds^i}(\frac{d}{ds}(\gamma(s)-s)-c_\infty^2\lambda_\infty^{-2}(s))||_{L^M} \lesssim \delta^2, |\mu(s)-\frac{2v_\infty s+y_\infty}{a_\infty+b_\infty s}| \lesssim \delta^2\langle s\rangle^{-\frac{3}{2}+\delta_1},\\
&\sup_{1\leq i\leq[\frac{N}{3}]} ||\langle s\rangle^{\frac{5}{2}-\delta_1}\frac{d^i}{ds^i}[\mu(s)-\frac{2v_\infty s+y_\infty}{a_\infty+b_\infty s}]||_{L^M} \lesssim \delta^2
\end{aligned}
\tag{3.32}
$$

The fact that we work with L^M instead of L^∞ for the derivatives is a technical complication due to the fact that we need a *compactness property* for the fixed point Theorem to apply, see below.

3.2. **Algebraic manipulations II; analysis of the radiation part.** We now look at $\begin{pmatrix} U \\ \bar{U}\end{pmatrix}$. As mentioned in the first section, we essentially break this into its root and dispersive part; more precisely, we first tweak this function a bit, after a careful analysis of the phase $(\Psi-\Psi_\infty)(s,y) = (\Psi-\tilde{\Psi}_\infty)(s,y)-s$.
From (3.13) we infer the relation[22]

$$
\begin{aligned}
&\Psi-\Psi_\infty(s,y)\\
&=\gamma(s)-s+y[\omega(s)\nu(s)-\frac{\beta(s)}{2}\nu(s)\lambda(s)(\mu_\infty-\mu)-\frac{a_\infty v_\infty-\frac{b_\infty y_\infty}{2}}{a_\infty+b_\infty s}]+\omega(s)\lambda(s)(\mu_\infty-\mu)(s)\\
&+\frac{b_\infty y^2}{4(a_\infty+b_\infty s)}-\frac{\beta}{4}[\nu y]^2-\frac{\beta}{4}[\lambda[\mu_\infty-\mu]]^2-[\frac{v_\infty^2 s a_\infty}{a_\infty+b_\infty s}-\frac{b_\infty v_\infty s y_\infty}{a_\infty+b_\infty s}+\gamma_\infty-\frac{b_\infty y_\infty^2}{4(a_\infty+sb_\infty)}]
\end{aligned}
$$

We decompose this into two parts, $(\Psi-\Psi_\infty)(s,y) = (\Psi-\Psi_\infty)_1(s)+(\Psi-\Psi_\infty)_2(s,y)$, where

$$
\begin{aligned}
&(\Psi-\Psi_\infty)_1(s)\\
&=\gamma(s)-s+\omega(s)\lambda(s)(\mu_\infty-\mu)(s)-\frac{\beta}{4}[\lambda[\mu_\infty-\mu]]^2-[\frac{v_\infty^2 s a_\infty}{a_\infty+b_\infty s}-\frac{b_\infty v_\infty s y_\infty}{a_\infty+b_\infty s}+\gamma_\infty-\frac{b_\infty y_\infty^2}{4(a_\infty+sb_\infty)}],
\end{aligned}
\tag{3.33}
$$

i. e. this is the part of $\Psi-\Psi_\infty$ which only depends on s and not on y. Then we define

$$
\begin{pmatrix} \tilde{U}(s,y) \\ \bar{\tilde{U}}(s,y)\end{pmatrix} := \begin{pmatrix} e^{-i(\Psi-\Psi_\infty)_1(s)}U(s,y+\lambda_\infty(\mu-\mu_\infty)(s)) \\ e^{+i(\Psi-\Psi_\infty)_1(s)}\bar{U}(s,y+\lambda_\infty(\mu-\mu_\infty)(s))\end{pmatrix}
$$

We decompose

$$
\begin{pmatrix} \tilde{U}(s,y) \\ \bar{\tilde{U}}(s,y)\end{pmatrix} = \sum_{i=1}^{6}\lambda_i\eta_{i,\mathrm{proper}} + \begin{pmatrix} \tilde{U}(s,y) \\ \bar{\tilde{U}}(s,y)\end{pmatrix}_{dis}
$$

We can then infer the parameters λ_i, $i=1,\ldots,5$ from the orthogonality condition (1.11), while the parameter λ_6 is governed by a suitable ODE. We now carefully analyze these equations. First, for $j=2,\ldots,6$, we have explicitly (recall (2.1) as well as (1.6))

$$
\langle\begin{pmatrix} e^{i(\Psi_\infty-\Psi)(t)}U(t,y+\lambda_\infty(\mu-\mu_\infty)(t)) \\ e^{-i(\Psi_\infty-\Psi)(t)}\bar{U}(t,y+\lambda_\infty(\mu-\mu_\infty)(t))\end{pmatrix}, \xi_{j,\mathrm{proper}}(\nu(t)y)\rangle = 0
$$

[22]Recall that we also defined $\mu_\infty(s) = \frac{2v_\infty s+y_\infty}{a_\infty+b_\infty s}$

This may be recast as[23]

$$\langle\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix},\xi_{j,\text{proper}}(\nu(t)y)\rangle=\langle\begin{pmatrix}e^{i(\Psi_\infty-\Psi)_1(t)}U(t,y+\lambda_\infty(\mu-\mu_\infty)(t))\\ e^{-i(\Psi_\infty-\Psi)_1(t)}\bar{U}(t,y+\lambda_\infty(\mu-\mu_\infty)(t))\end{pmatrix},\xi_{j,\text{proper}}(\nu(t)y)\rangle$$
$$=\langle\begin{pmatrix}e^{i(\Psi_\infty-\Psi)_1(t)}(1-e^{i(\Psi_\infty-\Psi)_2(t)}) & 0\\ 0 & e^{-i(\Psi_\infty-\Psi)_1(t)}(1-e^{-i(\Psi_\infty-\Psi)_2(t)})\end{pmatrix}$$
$$\begin{pmatrix}U(t,y+\lambda_\infty(\mu-\mu_\infty)(t))\\ \bar{U}(t,y+\lambda_\infty(\mu-\mu_\infty)(t))\end{pmatrix},\xi_{j,\text{proper}}(\nu(t)y)\rangle$$

The intuition here, to be made precise below, is that $(\Psi-\Psi_\infty)_2(s,y)$, when localized in y, decays quite rapidly in s. Our first task is filtering out the λ_i, $i=1,\dots,5$ from this relation, while avoiding λ_6 if possible. From the above we have

$$\sum_{i=1}^{5}\lambda_i\langle\eta_{i,\text{proper}},\xi_{l,\text{proper}}(\nu(t)y)\rangle+\lambda_6\langle\eta_{6,\text{proper}},\xi_{l,\text{proper}}(\nu(t)y)\rangle+\langle\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis},\xi_{l,\text{proper}}(\nu(t)y)\rangle$$
$$=\langle\begin{pmatrix}e^{i(\Psi_\infty-\Psi)_1(t)}(1-e^{i(\Psi_\infty-\Psi)_2(t)}) & 0\\ 0 & e^{-i(\Psi_\infty-\Psi)_1(t)}(1-e^{-i(\Psi_\infty-\Psi)_2(t)})\end{pmatrix}$$
$$\begin{pmatrix}U(t,y+\lambda_\infty(\mu-\mu_\infty)(t))\\ \bar{U}(t,y+\lambda_\infty(\mu-\mu_\infty)(t))\end{pmatrix},\xi_{l,\text{proper}}(\nu(t)y)\rangle,$$

where $l=2,\dots,6$. Commence with the case $l=2$. Observe that

$$\langle\eta_{6,\text{proper}},\xi_{2,\text{proper}}(\nu(t)y)\rangle=0=\langle\eta_{2,\text{proper}},\xi_{2,\text{proper}}(\nu(t)y)\rangle$$

Hence

$$\sum_{i=1}^{5}\lambda_i\langle\eta_{i,\text{proper}},\xi_{2,\text{proper}}(\nu(t)y)\rangle+\lambda_6\langle\eta_{6,\text{proper}},\xi_{2,\text{proper}}(\nu(t)y)\rangle=\sum_{i\neq 2,6}\lambda_i\langle\eta_{i,\text{proper}},\xi_{2,\text{proper}}(\nu(t)y)\rangle$$

Next, we observe that

$$\langle\eta_{6,\text{proper}},\xi_{3,\text{proper}}(\nu(t)y)\rangle=0=\langle\eta_{2,\text{proper}},\xi_{3,\text{proper}}(\nu(t)y)\rangle$$

Thus we have

$$\sum_{i=1}^{5}\lambda_i\langle\eta_{i,\text{proper}},\xi_{3,\text{proper}}(\nu(t)y)\rangle+\lambda_6\langle\eta_{6,\text{proper}},\xi_{3,\text{proper}}(\nu(t)y)\rangle=\sum_{i\neq 2,6}\lambda_i\langle\eta_{i,\text{proper}},\xi_{3,\text{proper}}(\nu(t)y)\rangle$$

Further, observe that for reasons of parity, we have

$$\langle\eta_{6,\text{proper}},\xi_{4,\text{proper}}(\nu(t)y)\rangle=0=\langle\eta_{2,\text{proper}},\xi_{4,\text{proper}}(\nu(t)y)\rangle$$

The conclusion is that

$$\sum_{i=1}^{5}\lambda_i\langle\eta_{i,\text{proper}},\xi_{4,\text{proper}}(\nu(t)y)\rangle+\lambda_6\langle\eta_{6,\text{proper}},\xi_{4,\text{proper}}(\nu(t)y)\rangle=\sum_{i\neq 2,6}\lambda_i\langle\eta_{i,\text{proper}},\xi_{4,\text{proper}}(\nu(t)y)\rangle$$

One concludes similarly for the inner product with $\xi_6(\nu(t)y)$. Next we consider the inner product with $\xi_{5,\text{proper}}(\nu(t)y)$. We note the following inner product relations:

$$\langle\eta_{1,\text{proper}},\xi_{5,\text{proper}}(\nu(t)y)\rangle=0,\ \langle\eta_{2,\text{proper}},\xi_{5,\text{proper}}(\nu(t)y)\rangle=a(t),\ \langle\eta_{3,\text{proper}},\xi_{5,\text{proper}}(\nu(t)y)\rangle=0$$
$$\langle\eta_{4,\text{proper}},\xi_{5,\text{proper}}(\nu(t)y)\rangle=0,\ \langle\eta_{5,\text{proper}},\xi_{5,\text{proper}}(\nu(t)y)\rangle=0,\ \langle\eta_{6,\text{proper}},\xi_{5,\text{proper}}(\nu(t)y)\rangle=b(t)$$

In the immediately preceding the function $a(t)$ can be forced to vanish nowhere upon choosing δ small enough. We don't need this information concerning $b(t)$. We can now infer the following relations: first

$$\lambda_2(t)=(\nu(t)-1)\langle\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis},\phi(t)\rangle+\sum_{i\neq 2}a_{2i}(t)\lambda_i(t),\tag{3.34}$$

[23] We also use the notation $(\Psi_\infty-\Psi)_1(s):=-(\Psi-\Psi_\infty)_1(s)$

where $\phi(t)$ denotes a certain time dependent vector-valued Schwartz function (with uniform decay estimates for all its derivatives, including its time derivatives), while the parameters $a_{2i}(t)$, $i\neq 2,6$ decay at the same rate as $\phi(t,x)(\Psi-\Psi_\infty)_2(t,x))$, for another Schwartz function $\phi(t,x)$ (we shall henceforth denote (vector valued) Schwartz functions (with respect to the 2nd argument) in this manner, without distinguishing between them, it being understood that they satisfy uniform-in-time decay estimates, including all their derivatives.)
In the same vein, the preceding calculations allow us to infer that the coefficients $\lambda_i(t)$, $i\neq 2,6$ satisfy the relations

$$\lambda_i(t)=(\nu(t)-1)\langle\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis},\phi(t)\rangle+\lambda_2(t)a_{i2}(t)+\lambda_6(t)a_{i6}(t),\tag{3.35}$$

where the coefficients $a_{i2}(t),a_{i6}(t)$ satisfy the same estimates as $a_{2i}(t)$(with $i\neq 2,6$) above. Of course if we substitute (3.34) here we can get rid of the 2nd term on the right (choosing δ small enough). We have used the fact that

$$\langle\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis},\xi_{l,\mathrm{proper}}(\nu(t).)\rangle=\langle\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis},\xi_{l,\mathrm{proper}}(\nu(t)y)-\xi_{l,\mathrm{proper}}(.)\rangle,\ l\neq 1$$

In order to complete the control of the root part, we thus need to finally consider $\lambda_6(t)$, which controls the contribution of the 'exotic mode'. This we filter out by means of

$$2\kappa_2\lambda_6(t)=\langle\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right),\xi_{1,\mathrm{proper}}\rangle$$

Upon differentiation, this relation implies the following:

$$\begin{aligned}&i2\kappa_2\dot{\lambda_6}(t)=\langle i\partial_t\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right),\xi_{1,\mathrm{proper}}\rangle\\ &=\langle\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right),\left(\begin{array}{cc}\partial_t[\Psi-\Psi_\infty]_1&0\\0&-\partial_t[\Psi-\Psi_\infty]_1\end{array}\right)\xi_{1,\mathrm{proper}}\rangle+i\langle\partial_t[\lambda_\infty(\mu-\mu_\infty)]\left(\begin{array}{c}\partial_x\tilde{U}\\ \partial_x\bar{\tilde{U}}\end{array}\right),\xi_{1,\mathrm{proper}}\rangle\\ &+\langle\left(\begin{array}{c}e^{-i(\Psi-\Psi_\infty)_1(t)}i\partial_tU(t,y+\lambda_\infty(\mu-\mu_\infty)(t))\\ -e^{i(\Psi-\Psi_\infty)_1(t)}i\partial_t\bar{U}(t,y+\lambda_\infty(\mu-\mu_\infty)(t))\end{array}\right),\xi_{1,\mathrm{proper}}\rangle\end{aligned}\tag{3.36}$$

We now carefully analyze each of the three expressions on the right. The key is to show that no quantity morally[24] of the form $(\nu(t)-1)^a\lambda_6(t)$, $(\nu(t)-1)^a\lambda_2(t)$, $a=1,2$, occurs, as this would sabotage any attempt at controlling λ_6 by means of ODE techniques, on account of the estimates (3.32). This appears to require a lot of careful bookkeeping: start with the first expression on the right. We have

$$\begin{aligned}&\langle\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right),\left(\begin{array}{cc}\partial_t[\Psi-\Psi_\infty]_1&0\\0&-\partial_t[\Psi-\Psi_\infty]_1\end{array}\right)\xi_{1,\mathrm{proper}}\rangle\\ &=\sum_{i\neq 2,6}\lambda_j(t)\langle\eta_{j,\mathrm{proper}},\left(\begin{array}{cc}\partial_t[\Psi-\Psi_\infty]_1&0\\0&-\partial_t[\Psi-\Psi_\infty]_1\end{array}\right)\xi_{1,\mathrm{proper}}\rangle\\ &+\langle\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis},\left(\begin{array}{cc}\partial_t[\Psi-\Psi_\infty]_1&0\\0&-\partial_t[\Psi-\Psi_\infty]_1\end{array}\right)\xi_{1,\mathrm{proper}}\rangle\end{aligned}$$

Next, write

$$\begin{aligned}&\langle\partial_t[\lambda_\infty(\mu-\mu_\infty)]\left(\begin{array}{c}\partial_x\tilde{U}\\ \partial_x\bar{\tilde{U}}\end{array}\right),\xi_{1,\mathrm{proper}}\rangle=\partial_t[\lambda_\infty(\mu-\mu_\infty)]\sum_{j=1}^{6}\lambda_j\langle\partial_x\eta_{j,\mathrm{proper}},\xi_{1,\mathrm{proper}}\rangle\\ &+\partial_t[\lambda_\infty(\mu-\mu_\infty)]\langle\left(\begin{array}{c}\partial_x\tilde{U}_{dis}\\ \partial_x\bar{\tilde{U}}_{dis}\end{array}\right),\xi_{1,\mathrm{proper}}\rangle\end{aligned}$$

[24]Observe that for example the quantity $\partial_t(\Psi-\Psi_\infty)(t,y)$ decays like $\nu(t)-1$.

Carefully observe from (3.32) that we get $|\partial_t[\lambda_\infty(\mu-\mu_\infty)](t)| \lesssim \langle t\rangle^{-\frac{3}{2}+\delta_1}$. Finally, consider the contribution of

$$\langle \begin{pmatrix} e^{-i(\Psi-\Psi_\infty)_1(t)} i\partial_t U(t, y+\lambda_\infty(\mu-\mu_\infty)(t)) \\ -e^{-i(\Psi-\Psi_\infty)_1(t)} i\partial_t U(t, y+\lambda_\infty(\mu-\mu_\infty)(t)) \end{pmatrix}, \xi_{1,\text{proper}}\rangle \tag{3.37}$$

This we reformulate using (3.14). Observe that we have

$$\begin{aligned}
&\begin{pmatrix} \partial_y^2 - 1 + 3\nu^2\phi_0^4(\nu(t)y) & 2\nu^2\phi_0^4(\nu(t)y) \\ -2\nu^2\phi_0^4(\nu(t)y) & -\partial_y^2 + 1 - 3\nu^2\phi_0^4(\nu(t)y) \end{pmatrix} \begin{pmatrix} e^{i(\Psi_\infty-\Psi)_1(t)} U(y+\lambda_\infty[\mu-\mu_\infty](t), t) \\ e^{-i(\Psi_\infty-\Psi)_1(t)} \bar{U}(y+\lambda_\infty[\mu-\mu_\infty](t), t) \end{pmatrix} \\
&= \begin{pmatrix} e^{i(\Psi_\infty-\Psi)_1(t)} & 0 \\ 0 & e^{-i(\Psi_\infty-\Psi)_1(t)} \end{pmatrix} \begin{pmatrix} \partial_y^2 - 1 + 3\nu^2\phi_0^4(\nu(t)y) & 2\nu^2 e^{2i(\Psi-\Psi_\infty)_1}\phi_0^4(\nu(t)y) \\ -2\nu^2 e^{-2i(\Psi-\Psi_\infty)_1}\phi_0^4(\nu(t)y) & -\partial_y^2 + 1 - 3\nu^2\phi_0^4(\nu(t)y) \end{pmatrix} \begin{pmatrix} U \\ \bar{U} \end{pmatrix}(.)
\end{aligned}$$

This shows that we can reformulate (3.37) as follows:

$$\begin{aligned}
&\langle \begin{pmatrix} e^{i(\Psi_\infty-\Psi)_1(t)} & 0 \\ 0 & e^{-i(\Psi_\infty-\Psi)_1(t)} \end{pmatrix} [i\partial_t + \mathcal{H}_1(t)](\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(y+\lambda_\infty[\mu-\mu_\infty](t), t)), \xi_{1,\text{proper}}\rangle \\
&- \langle \begin{pmatrix} e^{i(\Psi_\infty-\Psi)_1(t)} U(y+\lambda_\infty[\mu-\mu_\infty](t), t) \\ e^{-i(\Psi_\infty-\Psi)_1(t)} \bar{U}(y+\lambda_\infty[\mu-\mu_\infty](t), t) \end{pmatrix}, \tilde{\mathcal{H}}(t)^*\xi_{1,\text{proper}}\rangle,
\end{aligned}$$

where

$$\tilde{\mathcal{H}}(t)^* = \begin{pmatrix} \partial_y^2 - 1 + 3\nu^2\phi_0^4(\nu(t)y) & -2\nu^2\phi_0^4(\nu(t)y) \\ +2\nu^2\phi_0^4(\nu(t)y) & -\partial_y^2 + 1 - 3\nu^2\phi_0^4(\nu(t)y) \end{pmatrix}$$

$$\mathcal{H}_1(t) = \begin{pmatrix} \partial_y^2 - 1 + 3\nu^2\phi_0^4(\nu(t)y) & 2\nu^2 e^{2i(\Psi-\Psi_\infty)_1}\phi_0^4(\nu(t)y) \\ -2\nu^2 e^{-2i(\Psi-\Psi_\infty)_1}\phi_0^4(\nu(t)y) & -\partial_y^2 + 1 - 3\nu^2\phi_0^4(\nu(t)y) \end{pmatrix}$$

Moreover, we have

$$\tilde{\mathcal{H}}(t)^*\xi_{1,\text{proper}} = \begin{pmatrix} 3\nu^2\phi_0^4(\nu(t)y) - 3\phi_0^4(y) & -2\nu^2\phi_0^4(\nu(t)y) + 2\phi_0^4 \\ -[-2\nu^2\phi_0^4(\nu(t)y) + 2\phi_0^4] & -[3\nu^2\phi_0^4(\nu(t)y) - 3\phi_0^4(y)] \end{pmatrix}\xi_{1,\text{proper}},$$

which is of the form $\begin{pmatrix} \alpha \\ -\alpha \end{pmatrix}$. This reveals that

$$\begin{aligned}
&\langle \begin{pmatrix} e^{i(\Psi_\infty-\Psi)_1(t)} U(y+\lambda_\infty[\mu-\mu_\infty](t), t) \\ e^{-i(\Psi_\infty-\Psi)_1(t)} \bar{U}(y+\lambda_\infty[\mu-\mu_\infty](t), t) \end{pmatrix}, \tilde{\mathcal{H}}(t)^*\xi_{1,\text{proper}}\rangle \\
&= \sum_{j\neq 2,6} \lambda_j(t)\langle \eta_{j,\text{proper}}, \tilde{\mathcal{H}}(t)^*\xi_{1,\text{proper}}\rangle + \langle \begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}, \tilde{\mathcal{H}}(t)^*\xi_{1,\text{proper}}\rangle
\end{aligned}$$

Also, note that $\tilde{\mathcal{H}}(t)^*\xi_{1,\text{proper}} = (\nu(t)-1)\phi(t,.)$ for a suitable (vector-valued) Schwartz function $\phi(t,.)$. We now need to carefully analyze the expression

$$\langle \begin{pmatrix} e^{i(\Psi_\infty-\Psi)_1(t)} & 0 \\ 0 & e^{-i(\Psi_\infty-\Psi)_1(t)} \end{pmatrix} ([i\partial_t + \mathcal{H}_1(t)] \begin{pmatrix} U \\ \bar{U} \end{pmatrix})(y+\lambda_\infty[\mu-\mu_\infty](t), t), \xi_{1,\text{proper}}\rangle$$

We reformulate this as[25]

$$\langle [i\partial_t + \tilde{\mathcal{H}}_1(t)](\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(y,t)), \begin{pmatrix} e^{-i(\Psi_\infty-\Psi)_1(t)} & 0 \\ 0 & e^{+i(\Psi_\infty-\Psi)_1(t)} \end{pmatrix} \xi_{1,\text{proper}}(y-\lambda_\infty[\mu-\mu_\infty](t))\rangle$$

and use (3.14), in which we schematically write the righthand side as $\dot{\pi}\partial_\pi W + N(U,\pi)$, where the first summand refers to those expressions which only involve the modulation parameters and their derivatives, and not (explicitly) the radiation. Then we can schematically rewrite the above as

$$\langle \dot{\pi}\partial_\pi W, \tilde{\xi}_{1,\text{proper}}\rangle + \langle N(U,\pi), \tilde{\xi}_{1,\text{proper}}\rangle + \langle (e^{i(\Psi-\Psi_\infty)_2(t)} - 1)\begin{pmatrix} U \\ \bar{U} \end{pmatrix}, \phi(t,.)\rangle, \tag{3.38}$$

[25]Here we define $\tilde{\mathcal{H}}_1(s)$ like $\mathcal{H}(s)$ but with $\Psi-\Psi_\infty$ replaced by $(\Psi-\Psi_\infty)_1$.

where we have introduced the notation

$$\left(\begin{array}{cc} e^{-i(\Psi_\infty-\Psi)_1(t)} & 0\\ 0 & e^{-i(\Psi-\Psi_\infty)_1(t)}\end{array}\right)\xi_{1,\text{proper}}(y-\lambda_\infty(\mu-\mu_\infty)(t))=\tilde{\xi}_{1,\text{proper}}$$

We now carefully analyze the first two expressions in (3.38), again in order to check that these don't implicitly contain expressions of the form $(\nu(t)-1)^a\lambda_2(t)$, $(\nu(t)-1)^a\lambda_6(t)$, $a=1,2$. The third expression in (3.38) turns out to be small, as we'll see later on.

Now expand the schematic expression $\langle\dot{\pi}\partial_\pi W,\tilde{\xi}_{1,\text{proper}}\rangle$, invoking (3.14). First one obtains

$$\langle(\dot{\lambda}\lambda^{-1}-\beta\nu^2)(\tilde{\eta}_2-\frac{\beta}{2}\tilde{\eta}_5+\omega\tilde{\eta}_4),\tilde{\xi}_{1,\text{proper}}\rangle$$

We note that the vectors $\tilde{\eta}_j$ appearing here carry the phases $e^{\pm i(\Psi-\Psi_\infty)}$. Thus by modifying them by errors of size $O((e^{i(\Psi_\infty-\Psi)_2}-1)\phi(t,.))$, we can replace these phases by $e^{\pm i(\Psi-\Psi_\infty)_1}$. By abuse of notation we shall refer to these vectors again as $\tilde{\eta}_j$. Then we potentially have $\langle\tilde{\eta}_2,\tilde{\xi}_{1,\text{proper}}\rangle\neq 0$. Using the 5th modulation equation, we recall that

$$4i\kappa_2(\dot{\lambda}\lambda^{-1}-\beta\nu^2)=-\langle N(U,\pi),\tilde{\xi}_5\rangle+\langle\left(\begin{array}{c}U\\ \bar{U}\end{array}\right),(i\partial_t+\mathcal{H}(t)^*)\tilde{\xi}_5\rangle,$$

where

$$\tilde{\xi}_5(t,y)=\left(\begin{array}{c} e^{i(\Psi-\Psi_\infty)(t,y)}\sqrt{\nu}(t)\phi_0(\nu(t)[y-\lambda_\infty(\mu-\mu_\infty)(t)]\\ e^{-i(\Psi-\Psi_\infty)(t)}\sqrt{\nu}(t)\phi_0(\nu(t)[y-\lambda_\infty(\mu-\mu_\infty)(t)]\end{array}\right)$$

We put $\tilde{\phi}(y):=y^2\phi_0(y)$ and calculate

$$\begin{aligned}
&i\partial_t\left(\begin{array}{c} e^{i(\Psi-\Psi_\infty)(t,y)}\sqrt{\nu}(t)\tilde{\phi}(\nu(t)[y-\lambda_\infty(\mu-\mu_\infty)(t)]\\ e^{-i(\Psi-\Psi_\infty)(t,y)}\sqrt{\nu}(t)\tilde{\phi}(\nu(t)[y-\lambda_\infty(\mu-\mu_\infty)(t)]\end{array}\right)\\
&=i\left(\begin{array}{c} i\partial_t(\Psi-\Psi_\infty)(t,y)e^{i(\Psi-\Psi_\infty)(t,y)}\sqrt{\nu}(t)\tilde{\phi}(\nu(t)[y-\lambda_\infty(\mu-\mu_\infty)(t)]\\ -i\partial_t(\Psi-\Psi_\infty)(t,y)e^{-i(\Psi-\Psi_\infty)(t,y)}\sqrt{\nu}(t)\tilde{\phi}(\nu(t)[y-\lambda_\infty(\mu-\mu_\infty)(t)]\end{array}\right)\\
&+i\sqrt{\nu}(t)[-\partial_t[\lambda(t)(\mu-\mu_\infty)(t)]+y\dot{\nu}(t)]\left(\begin{array}{c} e^{i(\Psi-\Psi_\infty)(t,y)}\nabla\phi_0(\nu(t)[y-\lambda_\infty(\mu-\mu_\infty)(t)]\\ e^{-i(\Psi-\Psi_\infty)(t,y)}\nabla\phi_0(\nu(t)[y-\lambda_\infty(\mu-\mu_\infty)(t)]\end{array}\right)\\
&+\frac{i\dot{\nu}(t)}{2\sqrt{\nu(t)}}\left(\begin{array}{c} e^{i(\Psi-\Psi_\infty)(t,y)}\sqrt{\nu}(t)\phi_0(\nu(t)[y-\lambda_\infty(\mu-\mu_\infty)(t)]\\ e^{-i(\Psi-\Psi_\infty)(t,y)}\sqrt{\nu}(t)\phi_0(\nu(t)[y-\lambda_\infty(\mu-\mu_\infty)(t)]\end{array}\right)
\end{aligned}$$

Further note that for a certain $k\neq 1$

$$\begin{aligned}
&\mathcal{H}(t)^*\tilde{\xi}_5\\
&=\left(\begin{array}{cc} \partial_y^2-1+3\nu^2\phi_0^4(\nu(t)[y+\lambda_\infty(\mu_\infty-\mu)]) & -2\nu^2\phi_0^4(\nu(t)[y+\lambda_\infty(\mu_\infty-\mu)])e^{2i(\Psi-\Psi_\infty)}\\ +2\nu^2\phi_0^4(\nu(t)[y+\lambda_\infty(\mu_\infty-\mu)])e^{-2i(\Psi-\Psi_\infty)} & -[\partial_y^2-1+3\nu^2\phi_0^4(\nu(t)[y+\lambda_\infty(\mu_\infty-\mu)])]\end{array}\right)\tilde{\xi}_5\\
&=\left(\begin{array}{cc}\nu^2-1 & 0\\ 0 & -(\nu^2-1)\end{array}\right)\tilde{\xi}_5+\nu^2\tilde{\xi}_k+O(\partial_y^{1,2}[\Psi-\Psi_\infty]\phi(t,.))
\end{aligned}$$

Then we observe that

$$\begin{aligned}
&\langle\left(\begin{array}{c}U\\ \bar{U}\end{array}\right),\left(\begin{array}{cc}\nu^2-1 & 0\\ 0 & -(\nu^2-1)\end{array}\right)\tilde{\xi}_5+\nu^2\tilde{\xi}_k\rangle=\sum_{j\neq 2,6}\lambda_j\langle\eta_{j,\text{proper}},\left(\begin{array}{cc}\nu^2-1 & 0\\ 0 & -(\nu^2-1)\end{array}\right)\xi_{5,\text{proper}}(\nu(t)y)\rangle\\
&+\langle\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis},\left(\begin{array}{cc}\nu^2-1 & 0\\ 0 & -(\nu^2-1)\end{array}\right)\xi_{5,\text{proper}}(\nu(t)y)\rangle+\langle(e^{i(\Psi-\Psi_\infty)_2(t,.)}-1)\left(\begin{array}{c}U\\ \bar{U}\end{array}\right),\phi(t,.)\rangle
\end{aligned}$$

Continue by observing that $\langle\tilde{\eta}_{5,4,1},\tilde{\xi}_{1,\text{proper}}\rangle=0$, provided we abuse notation and change the phase in $\tilde{\eta}_i$ to $(\Psi-\Psi_\infty)_1$, which generates errors of the type $O((e^{i(\Psi-\Psi_\infty)_2(t)}-1)\phi(t,.))$. Continuing in this fashion, we note that $\langle\tilde{\eta}_3,\tilde{\xi}_{1,\text{proper}}\rangle\neq 0$ generically (but this function will only be of size $O((\nu(t)-1))$. From the 3rd modulation equation we get

$$-i\kappa_1\nu(\dot{\mu}\lambda_\infty-2\omega\nu)=-\langle N(U,\pi),\tilde{\xi}_4\rangle+\langle\left(\begin{array}{c}U\\ \bar{U}\end{array}\right),(i\partial_s+\mathcal{H}(s)^*)\tilde{\xi}_4\rangle$$

Write as before

$$\partial_s\tilde{\xi}_4 = \begin{pmatrix} i\partial_s[\Psi-\Psi_\infty](s) & 0 \\ 0 & -i\partial_s[\Psi-\Psi_\infty](s) \end{pmatrix}\tilde{\xi}_4 + [\dot{\nu}y + \partial_s[\lambda(\mu_\infty-\mu)]]\partial_x\tilde{\xi}_4$$
$$+ O(\dot{\nu}\phi(t,.))$$

Moreover, we have (for a suitable $j \neq 1$)

$$\begin{aligned} &\mathcal{H}(s)^*\tilde{\xi}_4 \\ &= \begin{pmatrix} \partial_y^2 - 1 + 3\nu^2(s)\phi_0^4(\nu(s)(y+\lambda_\infty(\mu_\infty-\mu))) & -2\nu^2\phi_0^4(\nu(s)(y+\lambda_\infty(\mu_\infty-\mu)))e^{2i(\Psi-\Psi_\infty)(s)} \\ +2\nu^2\phi_0^4(\nu(s)(y+\lambda_\infty(\mu_\infty-\mu)))e^{-2i(\Psi-\Psi_\infty)(s)} & -(\partial_y^2 - 1 + 3\nu^2(s)\phi_0^4(\nu(s)(y+\lambda_\infty(\mu_\infty-\mu))) \end{pmatrix}\tilde{\xi}_4 \\ &= \begin{pmatrix} \nu^2-1 & 0 \\ 0 & -(\nu^2-1) \end{pmatrix}\tilde{\xi}_4 + \nu^2\tilde{\xi}_j + O(\partial_y^{1,2}[\Psi-\Psi_\infty]_2\phi(t,.)) \end{aligned}$$

Thus we obtain

$$\langle\mathcal{H}(s)^*\tilde{\xi}_4, \begin{pmatrix} U \\ \bar{U} \end{pmatrix}\rangle = \sum_{j\neq 2,6}\lambda_j\langle\eta_{j,\text{proper}}, \begin{pmatrix} \nu^2-1 & 0 \\ 0 & -(\nu^2-1) \end{pmatrix}\xi_{4,\text{proper}}(\nu(s)y)\rangle$$
$$+ \langle\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}, \begin{pmatrix} \nu^2-1 & 0 \\ 0 & -(\nu^2-1) \end{pmatrix}\xi_{4,\text{proper}}(\nu(s)y)\rangle + O(\partial_y^{1,2}[\Psi-\Psi_\infty]_2\phi(t,.))$$
$$+ O((e^{i(\Psi-\Psi_\infty)_2}-1)\phi(t,.))$$

The preceding observations allow us to control the expression $\langle\dot{\pi}\partial_\pi W, \tilde{\xi}_{1,\text{proper}}\rangle$ in (3.38). Observe that in the preceding we also generated the (schematic) terms $(\nu(t)-1)\langle N(U,\pi), \tilde{\xi}_4\rangle$, $(\nu(t)-1)\langle N(U,\pi), \tilde{\xi}_5\rangle$. Now consider the terms at least quadratic with respect to the radiation in (3.38), i. e. the expression $\langle N(U,\pi), \tilde{\xi}_{1,\text{proper}}\rangle$. We are predominantly concerned with the quadratic contribution, which we spell out explicitly:

$$\langle\begin{pmatrix} -3U^2\tilde{\phi}_0^3 e^{i(\Psi_\infty-\Psi)}\nu^{\frac{3}{2}} - 6|U|^2 e^{i(\Psi-\Psi_\infty)}\nu^{\frac{3}{2}}\tilde{\phi}_0^3 - \bar{U}^2 e^{3i(\Psi-\Psi_\infty)}\nu^{\frac{3}{2}}\tilde{\phi}_0^3 \\ 3\bar{U}^2\tilde{\phi}_0^3 e^{-i(\Psi_\infty-\Psi)}\nu^{\frac{3}{2}} + 6|U|^2 e^{-i(\Psi-\Psi_\infty)}\nu^{\frac{3}{2}}\tilde{\phi}_0^3 - U^2 e^{-3i(\Psi-\Psi_\infty)}\nu^{\frac{3}{2}}\tilde{\phi}_0^3 \end{pmatrix}, \tilde{\xi}_{1,\text{proper}}\rangle \tag{3.39}$$

where we recall $\tilde{\phi}_0$ is ϕ_0 evaluated at $\nu(s)y + \lambda(\mu_\infty - \mu)$. Now we substitute

$$\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix} = \sum_{i=1}^{5}\lambda_i\eta_{i,\text{proper}} + \lambda_6\eta_{6,\text{proper}} + \begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}$$

Use (3.34), to reformulate this as[26]

$$\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}(t,.) = \sum_{i\neq 2,6}\lambda_i(t)(\eta_{i,\text{proper}} + a_{2i}(t)\eta_{2,\text{proper}}) + \lambda_6(t)(\eta_{6,\text{proper}} + a_{26}(t)\eta_{2,\text{proper}})$$
$$+ \begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}(t,.) + (\nu(t)-1)\phi(t,.)\langle\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}(t,.), \phi(t,.)\rangle$$

Back to (3.39), we first rewrite

$$\langle\begin{pmatrix} -U^2\tilde{\phi}_0^3 e^{i(\Psi_\infty-\Psi)(t,.)}\nu^{\frac{3}{2}} \\ \overline{U^2}\tilde{\phi}_0^3 e^{-i(\Psi_\infty-\Psi)(t,.)}\nu^{\frac{3}{2}} \end{pmatrix}, \tilde{\xi}_{1,\text{proper}}\rangle = \langle\begin{pmatrix} -\tilde{U}^2\phi_0^3(\nu.)\nu^{\frac{3}{2}} \\ \bar{\tilde{U}}^2\phi_0^3(\nu.)\nu^{\frac{3}{2}} \end{pmatrix}, \xi_{1,\text{proper}}\rangle + O((e^{i(\Psi-\Psi_\infty)_2(t,.)}-1)\phi(t,.))$$

---

[26] As usual, $\phi(t,.)$ represents various Schwartz functions, which in addition to all their derivatives, both with respect to t and x, satisfy uniform decay estimates. Also, $\partial_t\phi$ is of size at most $\dot{\nu}$.

Then note that schematically

$$\frac{1}{2}\langle\left(\begin{array}{c} -\tilde{U}^2\phi_0^3(\nu.)\nu^{\frac{3}{2}} \\ \overline{\tilde{U}}^2\phi_0^3(\nu.)\nu^{\frac{3}{2}} \end{array}\right), \xi_{1,\text{proper}}\rangle = \nu^{\frac{3}{2}}\Im\langle\tilde{U}_{dis}^2\phi_0^3(\nu.), \phi_0\rangle$$
$$+ 2\lambda_6[\sum_{i\neq 2,6}\lambda_i\nu^{\frac{3}{2}}\Im\langle\eta_{i,\text{proper}}^1 + a_{2i}\eta_{2,\text{proper}}, \phi_0^4(\nu.)(\eta_{6,\text{proper}} + a_{26}(t)\eta_{2,\text{proper}})^1\rangle$$
$$+ \Im\langle\phi(t,.), \tilde{U}_{dis}\rangle] + \sum_{i,j} a_{ij}(t)\lambda_i\lambda_j(t) + \sum_{j=1,2}(\nu(t)-1)^j\langle\tilde{U}_{dis}, \phi_1(t,.)\rangle\langle\tilde{U}_{dis}, \phi_2(t,.)\rangle$$

The expression

$$\langle\left(\begin{array}{c} -\overline{\tilde{U}}^2 e^{3i(\Psi-\Psi_\infty)}\phi_0^3(\nu(t)y)\nu^{\frac{3}{2}} \\ \tilde{U}^2 e^{-3i(\Psi-\Psi_\infty)}\phi_0^3(\nu(t)y)\nu^{\frac{3}{2}} \end{array}\right), \tilde{\xi}_{1,\text{proper}}\rangle$$

is handled similarly. Moreover, it is easily seen that

$$\langle\left(\begin{array}{c} -|\tilde{U}|^2\nu^{\frac{3}{2}}\phi_0^3(\nu y) \\ |\tilde{U}|^2\nu^{\frac{3}{2}}\phi_0^3(\nu y) \end{array}\right), \xi_{1,\text{proper}}\rangle = 0$$

Finally, we can summarize the discussion following (3.36) in the following schematic equality[27]:

$$\dot{\lambda}_6 = \lambda_6[\langle\left(\begin{array}{c} \tilde{U} \\ \bar{\tilde{U}} \end{array}\right)_{dis}, \phi\rangle + (\nu-1)\langle\left(\begin{array}{c} \tilde{U} \\ \bar{\tilde{U}} \end{array}\right)_{dis}, \phi\rangle + \dot{\nu} + \lambda_6(\nu-1) + \langle\phi(e^{i(\Psi-\Psi_\infty)_2}-1), \psi\rangle]$$
$$+ \langle N(\tilde{U}_{dis}^2, \pi), \phi_1\rangle + (\nu-1)\langle N(\tilde{U}_{dis}^2, \pi), \phi_2\rangle + \sum_{a=1,2}(\nu-1)^a\langle\left(\begin{array}{c} \tilde{U} \\ \bar{\tilde{U}} \end{array}\right)_{dis}, \phi\rangle + \dot{\nu}\langle\left(\begin{array}{c} \tilde{U} \\ \bar{\tilde{U}} \end{array}\right)_{dis}, \phi\rangle \tag{3.40}$$
$$+ (\nu-1)\langle\left(\begin{array}{c} \tilde{U} \\ \bar{\tilde{U}} \end{array}\right)_{dis}, \phi_1\rangle\langle\left(\begin{array}{c} \tilde{U} \\ \bar{\tilde{U}} \end{array}\right)_{dis}, \phi_2\rangle + O(\langle U^3 + U^5 + [(\nu-1)^2 + \dot{\nu}]U^2, \phi\rangle)$$

As usual the functions ϕ, ψ etc represent Schwartz functions (with respect to the spatial variable) with uniform decay estimates in time. One easily checks that all these functions have time derivatives decaying like $\dot{\nu}$. In the arguments below, we shall omit the time dependence, as one easily checks that any additional terms generated by this additional time dependence of the ϕ etc (for example when performing integrations by parts in t) can be handled by exactly the same methods or are much simpler to estimate. We now impose the condition $\lim_{s\to+\infty}\lambda_6(s) = 0$. Introducing the integrating factor

$$\Lambda(t) = \int_t^\infty\langle\left(\begin{array}{c} \tilde{U}(s,.) \\ \bar{\tilde{U}}(s,.) \end{array}\right)_{dis}, \phi\rangle + (\nu-1)(s)\langle\left(\begin{array}{c} \tilde{U}(s,.) \\ \bar{\tilde{U}}(s,.) \end{array}\right)_{dis}, \phi\rangle + \dot{\nu}(s) + \lambda_6(s)(\nu(s)-1)$$
$$+ \langle\phi(e^{i(\Psi-\Psi_\infty)_2(s,.)}-1), \psi\rangle ds,$$

this leads to the following relation:

$$\lambda_6(s) = -e^{-\Lambda(t)}\int_t^\infty e^{\Lambda(s)}[...]ds, \tag{3.41}$$

where [...] stands for the part on the righthand side of (3.40) without the expression $\lambda_6[...]$. The equations (3.34), (3.35), (3.41) completely govern the evolution of the root part $\left(\begin{array}{c} \tilde{U} \\ \bar{\tilde{U}} \end{array}\right)_{root}$. Thus, to conclude the discussion of the radiation part, we need to describe the evolution of $\left(\begin{array}{c} \tilde{U} \\ \bar{\tilde{U}} \end{array}\right)_{dis}$. This is straightforward

[27] The first instance of $\langle N(\tilde{U}_{dis}^2, \pi), \phi_1\rangle$ refers to the symplectic form $\langle\tilde{U}_{dis}^2 - \overline{\tilde{U}_{dis}}^2, \phi_1\rangle$.

from Duhamel's principle. Recall from Theorem 1.3 that we need to match the initial data $\begin{pmatrix} U(0,.) \\ \overline{U(0,.)} \end{pmatrix}_{dis} = \begin{pmatrix} A(.) \\ \bar{A}(.) \end{pmatrix}$. To this end write

$$\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(0,.) = \sum_{i=1}^{6} \alpha_i \eta_{i,\text{proper}} + \begin{pmatrix} A \\ \bar{A} \end{pmatrix} \tag{3.42}$$

The coefficients α_i here can be inferred from the orthogonality relations (2.4). Thus schematically[28] we get $\alpha_i = \langle \begin{pmatrix} U \\ \bar{U} \end{pmatrix}(0,.), \xi_{k(i),\text{proper}}\rangle = \langle \begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}(0,.), \tilde{\xi}_{k(i),\text{proper}}\rangle$. Using our standard decomposition we now get

$$\begin{aligned}
\alpha_i &= \langle \begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}(0,.), \tilde{\xi}_{k(i),\text{proper}}(0,.)\rangle \\
&= \sum_j \lambda_j(0)\langle \eta_{j,\text{proper}}, \tilde{\xi}_{k(i),\text{proper}}(0,.)\rangle + \langle \begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}(0,.), \tilde{\xi}_{k(i),\text{proper}}(0,.) - \xi_{k(i),\text{proper}}\rangle \\
&= \sum_j \lambda_j(0)\langle \eta_{j,\text{proper}}, \tilde{\xi}_{k(i),\text{proper}}(0,.)\rangle + (\nu(0)-1)\langle \begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}(0,.), \phi\rangle
\end{aligned}$$

Now, to prescribe the evolution of $\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}$, we refer to (1.8). Introduce the function

$$\begin{pmatrix} \tilde{U}^{(t)}(s,y) \\ \bar{\tilde{U}}^{(t)}(s,y) \end{pmatrix} := \begin{pmatrix} e^{-i(\Psi-\Psi_\infty)_1(t)} U(s, y+\lambda_\infty(\mu-\mu_\infty)(t)) \\ e^{+i(\Psi-\Psi_\infty)_1(t)} \bar{U}(s, y+\lambda_\infty(\mu-\mu_\infty)(t)) \end{pmatrix}, \begin{pmatrix} \tilde{U}^{(t)}(t,y) \\ \bar{\tilde{U}}^{(t)}(t,y) \end{pmatrix} = \begin{pmatrix} \tilde{U}(t,y) \\ \bar{\tilde{U}}(t,y) \end{pmatrix}, \tag{3.43}$$

Then we deduce the following equation:

$$\begin{aligned}
&[i\partial_s + \begin{pmatrix} \partial_y^2 - 1 + 3\phi_0^4 & 2\phi_0^4 \\ -2\phi_0^4 & -\partial_y^2 + 1 - 3\phi_0^4 \end{pmatrix}] \begin{pmatrix} \tilde{U}^{(t)} \\ \overline{\tilde{U}^{(t)}} \end{pmatrix}(s,y) = \begin{pmatrix} e^{-i(\Psi-\Psi_\infty)_1(t)} & 0 \\ 0 & e^{i(\Psi-\Psi_\infty)_1(t)} \end{pmatrix}[...] \\
&+ 2\begin{pmatrix} 0 & -e^{2i(\Psi-\Psi_\infty)_1(s)-2i(\Psi-\Psi_\infty)_1(t)} + 1 \\ e^{-2i(\Psi-\Psi_\infty)_1(s)+2i(\Psi-\Psi_\infty)_1(t)} - 1 & 0 \end{pmatrix} \phi_0^4 \begin{pmatrix} \tilde{U}^{(t)}(s) \\ \overline{\tilde{U}^{(t)}(s)} \end{pmatrix} \\
&+ O([\nu^2-1]\phi_0^4 U) + O([\mu_\infty - \mu](s)\lambda(s)\phi_0^4 U) + O([e^{i(\Psi-\Psi_\infty)_2(s,.)-1}]\phi_0^4 U)
\end{aligned} \tag{3.44}$$

The quantity [...] on the righthand side refers to the righthand expression in (1.8) translated by the amount $+\lambda_\infty(\mu-\mu_\infty)(t)$ in the spatial variable, but one uses the identifications

$$4i\kappa_2(\dot{\lambda}\lambda^{-1} - \beta\nu^2) = -\langle N, \tilde{\xi}_5\rangle + \langle U, (i\partial_s + \mathcal{H}(s)^*)\tilde{\xi}_5\rangle \text{ etc},$$

coming from the modulation equations, within[29] $\dot{\pi}\partial_\pi W$; thus we replace the left hand expressions by the ones on the right. We then project the preceding equation onto the dispersive part, and invoke Duhamel's principle, which results in the following equation governing the evolution of $\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}$:

$$\begin{aligned}
\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}(t,.) =& e^{it\mathcal{H}} P_s[\begin{pmatrix} e^{i(\Psi_\infty-\Psi)_1(t)} & 0 \\ 0 & e^{-i(\Psi_\infty-\Psi)_1(t)} \end{pmatrix}[\begin{pmatrix} A(.+\lambda_\infty(\mu-\mu_\infty)(t)) \\ \bar{A}(.+\lambda_\infty(\mu-\mu_\infty)(t)) \end{pmatrix} \\
&+ \sum_{j=1}^{6} \alpha_j[\eta_{j,\text{proper}}(.+\lambda_\infty(\mu-\mu_\infty)(t))]]] - i\int_0^t e^{i(t-s)\mathcal{H}}[...]_{dis}(s,.)ds,
\end{aligned} \tag{3.45}$$

[28]We really get a linear combination of expressions of the indicated form.

[29]Recall that we use the schematic notation $(i\partial_s + \mathcal{H}(s))\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(s,.) = \dot{\pi}\partial_\pi W + N(U,\pi)$.

in which [...] refers to the righthand side of (3.44). Also, the coefficients α_i are given by the formula detailed further above, i. e.

$$\alpha_i = \sum_j \lambda_j(0)\langle \eta_{j,\text{proper}}, \tilde{\xi}_{k(i),\text{proper}}(0,.)\rangle + (\nu(0)-1)\langle \begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}(0,.), \phi\rangle \tag{3.46}$$

Summary: *The coefficients* $\lambda_i(t)$, $i \neq 6$, *are given by the relation* (3.34), (3.35), *while the coefficient* $\lambda_6(t)$ *which controls the 'exotic mode' is given by the formula* (3.41). *In particular, the latter forces a value for* $\lambda_6(0)$. *The dispersive part of the 'tweaked radiation part', namely* $\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}$, *is governed by* (3.45), *upon fixing the condition* $\begin{pmatrix} U(0,.) \\ \bar{U}(0,.) \end{pmatrix}_{dis} = \begin{pmatrix} A(.) \\ \bar{A}(.) \end{pmatrix}$.

To conclude this subsection, we still need to **specify the estimates to be satisfied by the radiation part**; as for the root part, we shall need

$$\sup_{0\leq k\leq [\frac{N}{2}]} ||\langle t\rangle^{2-4\delta_1}\partial_t^k \lambda_i(t)||_{L^M} \lesssim \delta^2, \tag{3.47}$$

where δ_i, δ, N, M are as in (3.32). The reason why we don't work with L^∞ is again the compactness property[30]. As for the dispersive part, let C_k be sufficiently rapidly[31] growing numbers, $1 \leq k \leq N$; we shall impose $25^N\delta_2 << \delta_1$. Then we need

$$\begin{aligned}
&\sup_{0\leq k\leq N}\sup_{3i+j\leq k}\sup_{s\geq 0} C_k^{-1}||\langle s\rangle^{\frac{1}{2}-25^k\delta_2}\partial_s^i\partial_y^j U(s,y)||_{L_s^M L_y^M} \lesssim \delta,\\
&\sup_{\phi\in\mathcal{A}}\sup_{0\leq k\leq N}\sup_{3i+j\leq k}\sup_{s\geq 0} C_k^{-1}||\langle s\rangle^{1-20^k\delta_2}\phi\partial_s^i\partial_y^j U(s,y)||_{L_s^M L_y^M} \lesssim \delta\\
&\sup_{\phi\in\mathcal{A}}\sup_{s\geq 0}||\langle s\rangle^{\frac{3}{2}-\delta_3}\phi U(s,.)||_{L_y^\infty} \lesssim \delta,\ \sup_{s\geq 0}||CU(s,y)||_{L_y^2} \lesssim \delta,\\
&\sup_{0\leq k\leq N}\sup_{3i+j\leq k}\sup_{s\geq 0} C_k^{-1}||\langle s\rangle^{-10^k\delta_2}\partial_s^i\partial_y^j U(s,y)||_{L_s^M L_y^2} \lesssim \delta,
\end{aligned} \tag{3.48}$$

where C refers to the standard **pseudo-conformal operator** $C = y + 2is\partial_y$. Also, we denote by $\mathcal{A}$ the set of all Schwartz functions satisfying

$$\sup_{i\leq 100N}\sup_{x\in\mathbf{R}} |\langle x\rangle^{100}\partial_x^i\phi(x)| < 1$$

4. The iterative step.

4.1. **Deducing Theorem 1.3 from a fixed point Theorem.** It is now straightforward with our setup to formulate the iterative[32] step: We commence with a tuple of functions and parameters, as follows:

$$\{\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}(.,.), \lambda_1(.),\dots,\lambda_6(.),\nu_1(.),\beta_1(.),\omega_1(.),\mu_1(.),\gamma_1(.)\},$$

The functions $\nu_1(s)$, $\beta_1(s)$ stand for $\nu(s)-1$, $(\beta\nu - b_\infty\lambda_\infty^{-1})(s)$, respectively, while the functions $\mu_1(s)$, $\omega_1(s)$, $\gamma_1(s)$ stand for $(\mu(s) - \frac{2v_\infty s + y_\infty}{a_\infty + b_\infty s})$, $(\omega - \frac{c_\infty}{\lambda_\infty})(s)$, $\gamma(s) - s - \int_0^s \frac{c_\infty^2}{\lambda_\infty^2(\lambda)}d\lambda$, respectively, see the discussion in the last section; we let $c_\infty = v_\infty a_\infty - \frac{b_\infty y_\infty}{2}$. Of course we require the orthogonality conditions $\langle \begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}, \xi_{i,\text{proper}}\rangle =$

[30]Of course we can recover L^∞ bounds for all derivatives with exception of the top derivative from this information. We avoid this distinction in order to simplify matters.

[31]As usual the necessary rate of growth can be inferred from the proof.

[32]However, we shall not be able to construct the solution by iteration alone.

0, $i = 1, \ldots, 6$. Moreover, the function $\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}$ is to satisfy the estimates (3.48), the functions $\lambda_i(t)$ are to satisfy the inequalities (3.47), while the functions ν_1, β_1 etc. are to satisfy the corresponding estimates in (3.32). Upon fixing the 'initial condition' $\begin{pmatrix} U \\ \bar{U} \end{pmatrix}_{dis}(0, .) = \begin{pmatrix} A \\ \bar{A} \end{pmatrix}(.)$ as in Theorem 1.3, we then construct a map T_A which associates another tuple, characterized by primes, i. e. we get a tuple $\{\begin{pmatrix} U' \\ \bar{U}' \end{pmatrix}_{dis}, \ldots\}$, as follows: first, we construct the 'parameters at infinity' by means of the following rules:

$$a_\infty = \frac{\lambda(0)}{1+\nu_1(0)},\ b_\infty = [\beta(0)\nu(0) - \beta_1(0)]a_\infty,\ c_\infty = [\omega(0) - \omega_1(0)]a_\infty,\ y_\infty = (\mu(0) - \mu_1(0))a_\infty \tag{4.1}$$

$$v_\infty = \frac{[c_\infty + \frac{b_\infty y_\infty}{2}]}{a_\infty}$$

Observe that the assumed bounds then imply the statements concerning these parameters in Theorem 1.3. Next, re-construct the original quantities λ, β etc. from the tuple in the obvious fashion. This then also defines $\Psi - \Psi_\infty$ etc, see the beginning of subsection 3.2, provided we choose γ_∞ in such fashion that $(\Psi - \Psi_\infty)_1(0) = 0$. In particular, we can reconstruct $\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(t, .) = \begin{pmatrix} e^{i(\Psi-\Psi_\infty)_1(t)} & 0 \\ 0 & e^{-i(\Psi-\Psi_\infty)_1(t)} \end{pmatrix} \begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}(t, . - \lambda_\infty(\mu - \mu_\infty)(t))$. Then we use (3.45) to construct $\begin{pmatrix} \tilde{U}' \\ \bar{\tilde{U}}' \end{pmatrix}_{dis}$; simply use the un-primed quantities for the right-hand side. Next, define λ_2' via the righthand side of (3.34), λ_i' via the righthand side of (3.35), and λ_6' via (3.41). We next turn to the modulation parameters: define ν_1' as the righthand side of (3.24), and β_1' via (3.25). We can then also define b_∞' and a_∞' by means of the corresponding relations in (4.1), which in turn defines $\lambda_\infty'(.)$. Further, put[33]

$$B'(s) = \exp(\int_0^s [\beta'\nu'^2 + \frac{1}{4i\kappa_2}E_5](\sigma)d\sigma),$$

where E_5 is defined with respect to the un-primed quantities. We shall show later that under suitable assumptions on the tuple $\{\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}, \ldots\}$, we have $B'(s)^{-1} = c'\lambda_\infty'^{-1}(s) + o(\frac{1}{s})$, for suitable c'. Then define ω' via the formula

$$\begin{aligned}\omega'(s) = c'\lambda_\infty'(s)^{-1}[\omega(0) + \int_0^\infty B'(\sigma)[(i\kappa_1)^{-1}E_3 + \frac{\beta}{2i\kappa_1}E_4](\sigma)d\sigma] \\ - c'\lambda_\infty'(s)^{-1}\int_s^\infty B'(\sigma)[(i\kappa_1)^{-1}E_3 + \frac{\beta}{2i\kappa_1}E_4](\sigma)d\sigma] + o(s^{-1}),\end{aligned}$$

see the discussion preceding (3.28). We can infer from this a number c_∞' with the property[34]

$$\omega(s)' = \frac{c_\infty'}{\lambda_\infty'(s)} + o(\frac{1}{s}),$$

whence we can define $\omega_1(s)' = \omega(s)' - c_\infty'\lambda_\infty'^{-1}(s)$. Continue by setting

$$\mu(s)' = \mu(0) + \int_0^s \lambda_\infty'(\sigma)^{-1}[2\omega'\nu' + (i\kappa_1\nu(\sigma))^{-1}E_4(\sigma)]d\sigma \tag{4.2}$$

Again, under suitable assumptions on the original tuple we shall be able to infer the existence of numbers v_∞', y_∞' with the property[35]

$$\mu(s)' = \frac{2v_\infty' s + y_\infty'}{a_\infty' + b_\infty' s} + o(\frac{1}{s}),$$

[33] The quantity $B'(s)$ *is not the derivative*, but the new $B(s)$.

[34] We shall verify later that this definition is indeed meaningful.

[35] See last footnote

whence we can define $\mu_1(s)' = \mu(s)' - \frac{2v'_\infty s + y'_\infty}{a'_\infty + b'_\infty s}$. Also, we shall have $c'_\infty = v'_\infty a'_\infty - \frac{b'_\infty y'_\infty}{2}$. Finally, put

$$\gamma(s)' = \gamma(0) + \int_0^s [\nu'^2(\sigma) - (2i\kappa_2)^{-1}E_6(\sigma) + \nu'^2(\sigma)\omega'^2(\sigma) - \frac{1}{i\kappa_1}\omega(\sigma)E_4(\sigma) - \frac{1}{2\kappa_2}(i\kappa_2)^{-1}E_2(\sigma)]d\sigma,$$

whence we can define $\gamma_1(s)' = \gamma(s)' - s - \int_0^s \frac{c'^2_\infty}{\lambda'^2_\infty}$.

We can now reduce the proof of Theorem 1.3 to locating a fixed point of the map T_A. This follows from the next Proposition; let the norm $|||.|||$ be defined as in definition 4.4:

Proposition 4.1. *Let $A : \mathbf{R} \to \mathbf{C}$ be a smooth function satisfying $|||A||| < \delta$ for suitably small $\delta > 0$, as well as the orthogonality conditions $\langle \begin{pmatrix} A \\ \bar{A} \end{pmatrix}, \xi_{i,proper}\rangle = 0$, $i = 1, \ldots, 6$. Let $\{\begin{pmatrix} \tilde{U} \\ \tilde{\bar{U}} \end{pmatrix}_{dis}(.,.), \ldots\}$ be a tuple as above[36] satisfying the fixed point property[37]*

$$T_A\{\begin{pmatrix} \tilde{U} \\ \tilde{\bar{U}} \end{pmatrix}_{dis}(.,.), \ldots\} = \{\begin{pmatrix} \tilde{U} \\ \tilde{\bar{U}} \end{pmatrix}_{dis}(.,.), \ldots\}$$

Also, assume that $\sup_{s\geq 0}||U(s,.)||_{L^2_x} \lesssim \delta$, $\sup_{s\geq 0} s^{\frac{1}{2}}||U(s,.)||_{L^\infty_x} \lesssim \delta$. Define[38]

$$\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(s,.)$$
$$= \begin{pmatrix} e^{i(\Psi-\Psi_\infty)_1(s)} & 0 \\ 0 & e^{-i(\Psi-\Psi_\infty)_1(s)} \end{pmatrix} [\begin{pmatrix} \tilde{U} \\ \tilde{\bar{U}} \end{pmatrix}_{dis} + \sum_{j=1}^6 \lambda_j \eta_{j,proper}](s, . - \lambda_\infty(\mu - \mu_\infty)(s)),$$

Then the function (with $\mathcal{T}_\infty$ as in (1.7))

$$\psi(t,x) = W(t,x) + \mathcal{T}_\infty^{-1}[e^{is}U(s,.)](t,x)$$

is a non-generic blow-up solution of (1.1) exploding at time $t_* = \frac{1}{a_\infty b_\infty}$ (recall $a_\infty, b_\infty \sim 1$). We have

$$W(t,x) = e^{i(\gamma(s(t)) + [\omega(x-\mu)](s(t))}e^{-i\frac{\beta}{4}[\lambda(x-\mu)(s(t))]^2}\sqrt{\lambda(s(t))}\phi_0([\lambda(x-\mu)](s(t)), 1),$$

where $s(t) = \frac{a_\infty t}{a_\infty^{-1} - b_\infty t}$. Finally, we have

$$\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(0,x) = \begin{pmatrix} A \\ \bar{A} \end{pmatrix}(x) + \sum_{i=1}^6 \tilde{\lambda}_i \eta_{i,proper}$$

for certain numbers $\tilde{\lambda}_i$ with $|\tilde{\lambda}_i| \lesssim \delta^2$.

Proof. To begin with, note that the modulation equations (3.17) etc are satisfied. We continue by verifying that $\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(s,y)$ satisfies (1.8). Thus define a function $\begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix}(s,y)$ by means of the inhomogeneous linear equation

$$i\partial_s \begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix} + \mathcal{H}(s)\begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix} = -i(\dot{\lambda}\lambda^{-1} - \beta\nu^2)(\tilde{\eta}_2 - \beta\tilde{\eta}_5/2 + \omega\tilde{\eta}_4) \tag{4.3}$$

$$+ \frac{i}{4}(\dot{\beta} + \beta^2\nu^2)\tilde{\eta}_5 + i(\nu^2 - \dot{\gamma} + \nu^2\omega^2)\tilde{\eta}_1 \tag{4.4}$$

$$- i(\dot{\omega} + \beta\omega\nu^2)\tilde{\eta}_4 - i\nu(\dot{\mu}\lambda_\infty - 2\nu\omega)(-\omega\tilde{\eta}_1 - \tilde{\eta}_3 + \beta\tilde{\eta}_4/2) + N(U,\pi), \tag{4.5}$$

$$\begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix}(0,.) = \begin{pmatrix} U \\ \bar{U} \end{pmatrix}(0,.) \tag{4.6}$$

[36]In particular satisfying all the above-specified estimates.

[37]In particular, we assume that the operation of T_A is well-defined on this tuple. We shall soon analyze where T_A is well-defined.

[38]The function $(\Psi - \Psi_\infty)_1(s)$ is given by (3.33).

Define $\begin{pmatrix} \tilde{\Omega} \\ \bar{\tilde{\Omega}} \end{pmatrix}(t,.) := \begin{pmatrix} e^{-i(\Psi-\Psi_\infty)_1(t)} & 0 \\ 0 & e^{+i(\Psi-\Psi_\infty)_1(t)} \end{pmatrix} \begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix}(t,.+\lambda_\infty(\mu-\mu_\infty)(t))$, and use a decomposition

$$\begin{pmatrix} \tilde{\Omega} \\ \bar{\tilde{\Omega}} \end{pmatrix}(t,.) = \begin{pmatrix} \tilde{\Omega} \\ \bar{\tilde{\Omega}} \end{pmatrix}_{dis}(t,.) + \sum_{j=1}^{6} \mu_j(t)\eta_{j,\text{proper}} \tag{4.7}$$

Now we deduce the equation

$$\begin{aligned}
\begin{pmatrix} \tilde{\Omega} \\ \bar{\tilde{\Omega}} \end{pmatrix}_{dis}(t,.) &= e^{it\mathcal{H}} \begin{pmatrix} \tilde{U}^{(t)} \\ \bar{\tilde{U}}^{(t)} \end{pmatrix}_{dis}(0,.) - i\int_0^t e^{i(t-s)\mathcal{H}}[...]_{dis} ds \\
&- 2i\int_0^t e^{i(t-s)\mathcal{H}}[\begin{pmatrix} 0 & -e^{+2i(\Psi-\Psi_\infty)_1(s)-2i(\Psi-\Psi_\infty)_1(t)}+1 \\ e^{-2i(\Psi-\Psi_\infty)_1(s)+2i(\Psi-\Psi_\infty)_1(t)}-1 & \end{pmatrix} \\
&\qquad\qquad \times \phi_0^4 \begin{pmatrix} \tilde{\Omega}^{(t)}(s,.) \\ \bar{\tilde{\Omega}}^{(t)}(s,.) \end{pmatrix}]_{dis} ds \\
&- i\int_0^t e^{i(t-s)\mathcal{H}}\widetilde{[...]}_{dis}(s) ds
\end{aligned}$$

where [...] refers to the righthand side of (4.3) translated by $+\lambda_\infty(\mu-\mu_\infty)(t)$ in the spatial variable and 'twisted' by $\begin{pmatrix} e^{i(\Psi_\infty-\Psi)_1(t)} & 0 \\ 0 & e^{i(\Psi_\infty-\Psi)_1(t)} \end{pmatrix}$, and we put

$$\begin{aligned}
&\widetilde{[...]} \\
&= \begin{pmatrix} e^{i(\Psi_\infty-\Psi)_1(t)} & 0 \\ 0 & e^{-i(\Psi_\infty-\Psi)_1(t)} \end{pmatrix} \\
&\times \begin{pmatrix} -3\nu^2(s)\tilde{\phi}_0^4(.-\lambda_\infty(\mu-\mu_\infty)(s)) + 3\tilde{\phi}_0^4 & 2\tilde{\phi}_0^4 e^{2i(\Psi-\Psi_\infty)_1(s)}(-e^{2i(\Psi-\Psi_\infty)_2(t)}+1) \\ -2\tilde{\phi}_0^4 e^{-2i(\Psi-\Psi_\infty)_1(s)}(-e^{-2i(\Psi-\Psi_\infty)_2(t)}+1) & 3\nu^2(s)\tilde{\phi}_0^4(.-\lambda_\infty(\mu-\mu_\infty)(s)) - 3\tilde{\phi}_0^4 \end{pmatrix} \\
&\qquad\qquad \times \begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix}(s,.+\lambda_\infty(\mu-\mu_\infty)(t))
\end{aligned}$$

and as before we put $\tilde{\phi}_0(.) = \phi(.+\lambda_\infty(\mu-\mu_\infty)(t))$. From the iterative step and the fact that the modulation equations are satisfied, we then deduce that

$$\begin{aligned}
&\begin{pmatrix} \tilde{\Omega}-\tilde{U} \\ \overline{\tilde{\Omega}-\tilde{U}} \end{pmatrix}_{dis}(t,.) \\
&= -2i\int_0^t e^{i(t-s)\mathcal{H}}[\begin{pmatrix} 0 & -e^{+2i(\Psi-\Psi_\infty)_1(s)-2i(\Psi-\Psi_\infty)_1(t)}+1 \\ e^{-2i(\Psi-\Psi_\infty)_1(s)+2i(\Psi-\Psi_\infty)_1(t)}-1 & \end{pmatrix} \\
&\qquad\qquad \times \phi_0^4 \begin{pmatrix} \tilde{\Omega}^{(t)}(s,.) - \tilde{U}^{(t)}(s,.) \\ \bar{\tilde{\Omega}}^{(t)}(s,.) - \bar{\tilde{U}}^{(t)}(s,.) \end{pmatrix}]_{dis} ds \\
&- i\int_0^t e^{i(t-s)\mathcal{H}}\widetilde{\Delta[...]}_{dis}(s) ds,
\end{aligned} \tag{4.8}$$

where the expression $\widetilde{\Delta[...]}_{dis}(s)$ is defined as above but with Ω replaced by the difference $\Omega - U$. Next, we have

$$i\partial_s\langle\begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix}, \tilde{\xi}_i\rangle = \langle(i\partial_s + \mathcal{H}(s))\begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix}, \tilde{\xi}_i\rangle - \langle\begin{pmatrix} \Omega \\ \bar{\Omega} \end{pmatrix}, (i\partial_s + \mathcal{H}^*(s))\tilde{\xi}_i\rangle,\ i = 2,\dots,6$$

whence we obtain

$$i\partial_s\langle\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}-\begin{pmatrix}U\\ \bar{U}\end{pmatrix},\tilde{\xi}_i\rangle=-\langle\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}-\begin{pmatrix}U\\ \bar{U}\end{pmatrix},(i\partial_s+\mathcal{H}^*(s))\tilde{\xi}_i\rangle,\ i=2,\dots,6$$

Thus we obtain from $(\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}-\begin{pmatrix}U\\ \bar{U}\end{pmatrix})(0,.)=0$ the relation

$$\begin{aligned}i\langle\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}-\begin{pmatrix}U\\ \bar{U}\end{pmatrix},\tilde{\xi}_i\rangle(t,.)&=\langle\begin{pmatrix}\tilde{\Omega}\\ \bar{\tilde{\Omega}}\end{pmatrix}-\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix},\Xi_i\rangle(t,.)\\ &=-\int_0^t\langle\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}-\begin{pmatrix}U\\ \bar{U}\end{pmatrix},(i\partial_s+\mathcal{H}^*(s))\tilde{\xi}_i\rangle(s)ds,\ i=2,\dots,6,\end{aligned}\tag{4.9}$$

where the first equality can be used to define Ξ_i in the obvious fashion. We shall use the last relation to solve for the coefficients μ_i, $i=1,\dots,5$. Finally, we consider the coefficient μ_6:

$$\begin{aligned}&i\dot{\mu}_6(t)=\langle i\partial_t\begin{pmatrix}\tilde{\Omega}\\ \bar{\tilde{\Omega}}\end{pmatrix},\xi_{1,\mathrm{proper}}\rangle\\ &=\langle\begin{pmatrix}\tilde{\Omega}\\ \bar{\tilde{\Omega}}\end{pmatrix},\begin{pmatrix}\partial_t(\Psi-\Psi_\infty)_1 & 0\\ 0 & -\partial_t(\Psi-\Psi_\infty)_1\end{pmatrix}\xi_{1,\mathrm{proper}}\rangle+\langle\partial_t(\lambda_\infty(\mu-\mu_\infty)(t))\begin{pmatrix}\partial_x\tilde{\Omega}\\ \overline{\partial_x\tilde{\Omega}}\end{pmatrix},\xi_{1,\mathrm{proper}}\rangle\\ &+\langle\begin{pmatrix}e^{-i(\Psi-\Psi_\infty)_1(t)}i\partial_t\tilde{\Omega}(t,y+\lambda_\infty(\mu-\mu_\infty)(t))\\ \overline{-e^{-i(\Psi-\Psi_\infty)_1(t)}i\partial_t\tilde{\Omega}(t,y+\lambda_\infty(\mu-\mu_\infty)(t))}\end{pmatrix},\xi_{1,\mathrm{proper}}\rangle\end{aligned}$$

Now we recall the corresponding identity in the derivation of (3.40), namely (3.36), take the difference of the latter and the identity above, and proceed as the paragraphs after (3.36). Using the fact that $\lambda_6(0)=\mu_6(0)$, we deduce the schematic identity

$$(\lambda_6-\mu_6)(t)=\int_0^t[\langle\begin{pmatrix}\tilde{\Omega}\\ \bar{\tilde{\Omega}}\end{pmatrix}(s,.)-\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}(s,.),\phi(s)\rangle+\langle\partial_x\begin{pmatrix}\tilde{\Omega}\\ \bar{\tilde{\Omega}}\end{pmatrix}(s,.)-\partial_x\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}(s,.),\psi(s)\rangle]ds\tag{4.10}$$

Now from (4.8), (4.9), (4.10), as well as (4.7) and the linear estimate Theorem 2.1, we easily deduce the estimate

$$\sup_{0\leq t\leq T}[||[\tilde{\Omega}-\tilde{U}]_{dis}(t,.)||_{H^1}+\sum_{i=1}^6|\lambda_i-\mu_i|(t)]\lesssim T\sup_{0\leq t\leq T}[||[\tilde{\Omega}-\tilde{U}]_{dis}(t,.)||_{H^1}+\sum_{i=1}^6|\lambda_i-\mu_i|(t)]$$

Now choose $T>0$ small enough to get the identity $\Omega|_{[0,T]}=U|_{[0,T]}$. Continuing in this fashion implies $U(.,.)=\Omega(.,.)$. Next, observe that the condition

$$\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}(0,.)=P_s[\sum_i\eta_{i,\mathrm{proper}}(.+\lambda_\infty(\mu-\mu_\infty)(0))\langle\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}(0,.),\tilde{\xi}_{k(i),\mathrm{proper}}\rangle+\begin{pmatrix}A\\ \bar{A}\end{pmatrix}(.+\lambda_\infty(\mu-\mu_\infty)(0))]\tag{4.11}$$

in addition to $P_{root}\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}(0,.)=\sum_{i=1}^6\lambda_i(0)\eta_{i,\mathrm{proper}}$ uniquely determines $\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}(0,.)$, hence in conjunction with the values of the modulation parameters also $\begin{pmatrix}U\\ \bar{U}\end{pmatrix}(0,.)$. Then one verifies that $\begin{pmatrix}U\\ \bar{U}\end{pmatrix}(0,.)=\begin{pmatrix}A\\ \bar{A}\end{pmatrix}+\sum_i\alpha_i\eta_{i,\mathrm{proper}}$ with the α_i defined as in (3.46) is consistent with (4.11), as well as the root part of $\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}(0,.)$.

Now reverse the algebraic manipulations that led to (1.8). We deduce that

$$Z(t,x)=W(t,x)+\mathcal{T}_\infty^{-1}[e^{is}U(s,.)](t,x)$$

is indeed a non-generic blow-up solution of (1.1). One checks that

$$\mathcal{T}_{\infty}^{-1}[e^{is}U(s,.)]$$
$$= (a_{\infty}^{-1}-b_{\infty}t)^{-\frac{1}{2}}e^{-i\frac{b_{\infty}x^2}{4(a_{\infty}^{-1}-b_{\infty}t)}}e^{i(\gamma_{\infty}+v_{\infty}x-v_{\infty}^2\frac{a_{\infty}t}{a_{\infty}^{-1}-b_{\infty}t}-v_{\infty}y_{\infty})}e^{i(\frac{a_{\infty}t}{a_{\infty}^{-1}-b_{\infty}t})}U(\frac{a_{\infty}t}{a_{\infty}^{-1}-b_{\infty}t},\frac{x-2v_{\infty}a_{\infty}t}{a_{\infty}^{-1}-b_{\infty}t}-y_{\infty})$$

The assumptions in the Proposition imply that this remains bounded with respect to L^{∞} as $t\to t_{*}=\frac{1}{a_{\infty}b_{\infty}}$, while due to the asymptotic relations (3.32) the principal soliton part $W(t,x)$ blows up according to the non-generic profile. Finally, recall the decomposition (3.42) in which we use (3.46). Our assumptions (3.32) as well as (3.47) imply the last statement of the Proposition. $\square$

4.2. **Deducing the fixed point from a priori estimates.** We now need to demonstrate the existence of a fixed point for the map T_A on the set of tuples satisfying the above specified inequalities. This will follow from an application of the Schauder-Tychonoff fixed point Theorem, which we recall here:

Theorem 4.2. *(Schauder-Tychonoff) A non-empty compact convex subset S of a Banach space has the fixed point property, i. e. for any continuous map $T:S\longrightarrow S$ there exists $x_T\in S$ satisfying $T(x_T)=x_T$.*

We now need to locate such a set S. We construct this as follows: first, for M,N as before ((3.32), (3.47)) and very large[39] $K>0$ introduce the norm[40]

$$\begin{aligned}&|||U|||_{S^{N,K}}\\ &=\sum_{0\le k\le N}C_k^{-1}\sum_{3i+j\le k}[\sup_{s\ge0}\langle s\rangle^{\frac12-25^k\delta_2}||\partial_s^i\partial_y^jU(s,y)||_{L_s^ML_y^M}+\sup_{s\ge0}\langle s\rangle^{-10^k\delta_2}||\partial_s^i\partial_y^jU(s,y)||_{L_s^ML_y^2}\\ &+\sup_{\phi\in\mathcal{A}}\sup_{s\ge0}\langle s\rangle^{1-20^k\delta_2}||\phi\partial_s^i\partial_y^jU(s,y)||_{L_s^ML_y^M}]+\sup_{s\ge0}[\sup_{\phi\in\mathcal{A}}\langle s\rangle^{\frac32-\delta_3}||\phi U(s,.)||_{L_y^{\infty}}+\sup_{s\ge0}||CU(s,y)||_{L_y^2}\\ &\qquad+\sum_{1\le k\le N-1}K^{-k}\sup_{3i+j=k}||C\partial_s^i\partial_y^jU(s,.)||_{L_s^ML_y^2}],\end{aligned}$$

where C is as in (3.48). The role of the last summand is to ensure uniform spatial decay on finite time intervals, again needed for compactness. Also, let $|||U|||_{S^N}$ be defined as above, but with the last summand replaced by $||CU||_{L_y^2}$, where we use $C=x+2is\partial_x$. Define the Banach spaces $S^{N,K}$, S^N, as the completion of $\mathcal{S}(\mathbf{R}^2)$, with respect to these norms. Now for a tuple $\Gamma:=\{\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis},\ldots\}$ as before, define the norm[41] (as usual we let $\langle s\rangle=|s|+1$)

$$\begin{aligned}|||\Gamma|||_{\tilde{S}^{N,K}}&:=|||\tilde{U}_{dis}|||_{S^{N,K}}+\delta^{-1}[\sup_{0\le s<\infty}\langle s\rangle^{\frac12-\delta_1}|\nu_1(s)|+\sum_{1\le k\le[\frac{N}{3}]}||\langle s\rangle^{\frac32-2\delta_1}\frac{d^k}{ds^k}\nu_1(s)||_{L^M}+\sup_{0\le s<\infty}\langle s\rangle^{\frac32-\delta_1}|\beta_1(s)|\\ &+\sum_{1\le k\le[\frac{N}{3}]}||\langle s\rangle^{2-2\delta_1}\frac{d^k}{ds^k}\beta_1(s)||_{L^M}+\sup_{0\le s<\infty}|\langle s\rangle^{\frac32-\delta_1}\omega_1(s)|+\sum_{1\le k\le[\frac{N}{3}]}||\langle s\rangle^{\frac32-2\delta_1}\frac{d^k}{ds^k}\omega_1(s)||_{L^M}\\ &+\sup_{0\le s<\infty}|\langle s\rangle^{\frac12-\delta_1}\frac{d}{ds}\gamma_1(s)|+\sum_{2\le k\le[\frac{N}{3}]}||\langle s\rangle^{\frac32-2\delta_1}\frac{d^k}{ds^k}\gamma_1(s)||_{L^M}+\sup_{0\le s<\infty}|\langle s\rangle^{\frac32-\delta_1}|\mu_1(s)|\\ &+\sum_{1\le k\le[\frac{N}{3}]}||\langle s\rangle^{\frac52-\delta_1}\frac{d^k}{ds^k}\mu_1(s)||_{L^M}+\sum_{i=1}^{6}\sum_{0\le k\le[\frac{N}{3}]}||\langle t\rangle^{2-4\delta_1}\frac{d^k}{dt^k}\lambda_i(t)||_{L^M}]\end{aligned}$$

[39]This parameter will eventually depend on a time T.

[40]We only include the parameters N,K as superscripts in the norm, since we shall only vary these.

[41]We use the notation $\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}=\begin{pmatrix}\tilde{U}_{dis}\\ \bar{\tilde{U}}_{dis}\end{pmatrix}$.

Also, let $|||\Gamma|||_{\tilde{S}^N}$ be defined as above but with $||\tilde{U}_{dis}||_{S^{N,K}}$ replaced by $||\tilde{U}_{dis}||_{S^N}$. Then we define the restrictions $|||.|||_{S^N([0,T))}$ etc for any time interval $[0,T)$ in the obvious fashion, and denote $S^N([0,T))$, $\tilde{S}^{N,K}([0,T))$ as completion of $\mathcal{S}([0,T)\times\mathbf{R})$ and of

$$\mathcal{S}([0,T)\times\mathbf{R})\times(C^\infty[0,T))^{11},$$

respectively, with respect to the above norms localized to $[0,T)\times\mathbf{R}$.Then we have[42]

Lemma 4.3. *Fix T, $\infty > T > 0$. For any $R\geq 0$, the set of tuples*[43] $A^{(0)}_{[0,T)} := \{\Gamma$ *on* $[0,T)\times\mathbf{R}\,|\,|||\Gamma|||_{\tilde{S}^{N,K}[0,T)}\leq R\delta\}$ *equipped with the norm* $|||.|||_{\tilde{S}^{N-4,K}[0,T)}$ *is a compact convex subset of*

$$A_{[0,T)} := \{\Gamma \text{ defined on } [0,T)\times\mathbf{R}\,|\,|||\Gamma|||_{\tilde{S}^{N-4,K}[0,T)}\leq R\delta\}$$

Proof. We demonstrate the compactness assertion: thus consider a sequence of tuples $\{\Gamma_i\}_{i\geq 1}\subset A^{(0)}_{[0,T)}$. Consider the functions $\begin{pmatrix}\tilde{U}_i\\ \bar{\tilde{U}}_i\end{pmatrix}_{dis} = \begin{pmatrix}\tilde{U}_{i,dis}\\ \bar{\tilde{U}}_{i,dis}\end{pmatrix}$. By assumption, letting $\phi_\rho(x) := \phi(\frac{x}{\rho})$, where $\phi(.)$ smoothly localizes to $|x|>1$, we have that

$$\lim_{\rho\to\infty}\sup_l\sum_{0\leq k\leq N-4}\sum_{3i+j\leq k}||\phi_\rho(x)\partial_t^i\partial_x^j\tilde{U}_{l,dis}||_{L_t^M L_x^2[0,T)} = 0$$

Indeed, this follows from uniform control over $||C\partial_t^i\partial_x^j\tilde{U}_{l,dis}||_{L_t^M L_x^2[0,T)}$. Combining this with the fact that $\sum_{0\leq k\leq N}\sup_{3i+j\leq k}||\partial_s^i\partial_x^j\tilde{U}_{i,dis}||_{L_s^M L_x^2[0,T)}$ is uniformly bounded and applying the Rellich-Kondrakhov compactness Theorem as well as Sobolev embedding, we obtain a subsequence (which we again label $\begin{pmatrix}\tilde{U}_i\\ \bar{\tilde{U}}_i\end{pmatrix}_{dis}$) which converges with respect to $\sum_{0\leq k\leq N-4}\sup_{3i+j\leq k}||\partial_s^i\partial_x^j(.)||_{L_s^M L_x^2[0,T)}$ as well as the remaining norms in $||.||_{S^{N-4,K}}$ to a limit[44] $\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}$. Passing to a further subsequence, we may assume that all $\partial_s^i\partial_x^j\tilde{U}_{i,dis}$, $N\geq 3i+j\geq N-4$, converge weakly[45] with respect to $L_s^M L_x^2[0,T)$, $L_s^M L_x^M[0,T)$, and one checks that the corresponding limits necessarily equal $\partial_s^i\partial_x^j\tilde{U}_{dis}$ in the distributive sense, respectively. Also, $|||\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}|||_{S^N}\leq R\delta$.

Now consider the root part, i. e. the functions $\lambda_{j,i}(t)$, $j=1,\ldots,6$. By assumption, we have a uniform bound on $\sum_{0\leq k\leq[\frac{N}{3}]}||\langle t\rangle^{2-4\delta_1}\frac{d^k}{dt^k}\lambda_{j,i}(t)||_{L^M[0,T)}$. By the Arzela-Ascoli Theorem, we can then choose a converging subsequence with respect to $\sum_{0\leq k\leq[\frac{N}{3}]-2}||\langle t\rangle^{2-4\delta_1}\frac{d^k}{dt^k}(.)||_{L^M[0,T)}$ whose limit satisfies the desired estimates. The argument for the modulation parameters $\nu_1(t)=\nu(t)-1$ etc. is identical. □

Now define the sets $A^{(n)}_{[0,T)}$, $n\geq 1$ inductively as follows: first, we can modify the inductive step T_A to the interval $[0,T)$, by simply replacing ∞ by T in the formulae for the modulation parameters and root parameters. By abuse of notation refer to this by T_A as well. Then put

$$A^{(n)}_{[0,T)} := \text{convhull}[A^{(0)}_{[0,T)}\cap\overline{(T_A A^{(n-1)}_{[0,T)})}]$$

The closure operation is always with respect to $|||.|||_{\tilde{S}^{N-4,K}}$. Then clearly $A^{(n)}_{[0,T)}\subset A^{(n-1)}_{[0,T)}$, and these are all compact convex subsets of $A_{[0,T)}$.

Everything now reduces to the following **core analytic Theorem**: first we make a definition:

---

[42]Recall that the first entry of a tuple is always required t also satisfy the orthogonality conditions

[43]We omit the dependence of these sets on R in the notation, it being understood that R below will be fixed throughout.

[44]Clearly this limit satisfies the same orthogonality relations, whence we may apply the subscript $_{dis}$.

[45]It is at this stage that we need L^M instead of L^∞

Definition 4.4. *We call a function $A(x) : \mathbf{R} \to \mathbf{C}$ admissible provided the estimate*

$$|||A||| := \sup_{0 \leq k \leq N} ||\langle x\rangle^{100} \frac{d^k}{dx^k} A(x)||_{L^1 \cap L^2} \leq \delta$$

is satisfied.

Theorem 4.5. *Let A be admissible. Let N satisfy the specifications in* (3.32), (3.48)*, and $\delta > 0$ small enough. There exists $R > 0$ sufficiently large such that with the corresponding $A^0_{[0,T)}$ etc constructed as above we have the following: For every $T > 0$, there exists a number $K = K(N,T)$, as well as an index $n_0(N)$ such that for $n \geq n_0$, we have $T_A A^{(n)}_{[0,T)} \subset A^{(n)}_{[0,T)}$; moreover, T_A acts continuously on $A^{(n)}_{[0,T)}$. The last assertions are always non-vacuous if $\delta > 0$ is sufficiently small, since then $A^{(n)}_{[0,T)} \neq \emptyset$. Thus by Theorem 4.2 there exists a tuple $\Gamma_T \in A^{(n)}_{[0,T)}$ with the property $T_A \Gamma_T = \Gamma_T$. For $T < \tilde{T}$, one has the inequality*

$$|||\Gamma_{\tilde{T}}|_{[0,T)\times\mathbf{R}}|||_{\tilde{S}^{N,K(N,T)}} \lesssim \delta$$

Also, we get the uniform[46] *bounds*

$$\sup_{T>s\geq 0} ||\tilde{U}_T(s,.)||_{L^2_x} \lesssim \delta, \ \sup_{T>s\geq 0} \langle s\rangle^{\frac{1}{2}} ||\tilde{U}_T(s,.)||_{L^\infty_x} \lesssim \delta$$

Assuming this, we can now conclude the following:

Theorem 4.6. *There exists a fixed point Γ for T_A (acting on $[0,\infty) \times \mathbf{R}$) satisfying the assumptions* (3.32)*,* (3.47)*,* (3.48)*. Thus the assumption of Proposition 4.1 is realizable.*

Proof. Let $T_i = i$, $i \geq 1$. Then construct fixed points Γ_i for the operation of $T_A|_{[0,T_i]}$ as in the preceding Theorem. Thanks to the uniform bounds for $|||\Gamma_i|_{[0,j)}|||_{\tilde{S}^{N,K(N,T_j)}}$, $j \leq i$, and invoking another compactness argument as before, we can select a subsequence $\Gamma_{j,i}$ which converges on $[0,T_j)$ with respect to $|||.|||_{\tilde{S}^{N-4,K(N,j)}}$. Observe that we only need a uniform bound on $K(N,T)$ for bounded T here, as we have arranged. Doing this for $j = 1, 2, \ldots$ and invoking the Cantor diagonal argument, we then construct a subsequence, which we again label Γ_i, which converges on every $[0,T_j]$, $j \geq 1$ to a tuple in $\tilde{S}^{N,K(N,j)}$. The limits then fit coherently to define a tuple Γ on $[0,\infty)$ living in $\tilde{S}^N$, which is the desired fixed point. $\square$

5. The proof of the core analytic estimates, Theorem 4.5.

We shall show that $T_A(A^{(n)}_{[0,T)}) \subset A^{(0)}_{[0,T)}$, provided n is large enough. The proof will also reveal the continuity of the operation T_A. Also, we shall show that if one iterates T_A starting with the tuple $\Gamma_{trivial} := \{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0, \ldots, \lambda(0), \beta(0), 0, 0, 0\}$, one always stays inside $A^{(0)}$ provided δ, δ_i etc are chosen suitably, whence $A^{(i)} \neq \emptyset \ \forall i \geq 0$. Then observe that

$$T_A(A^{(n+1)}_{[0,T)}) \subset A^{(0)}_{[0,T)}$$

as well as

$$T_A(\text{convhull}(\overline{T_A A^{(n)}_{[0,T)}})) \subset T_A(\text{convhull}(\overline{T_A A^{(n-1)}_{[0,T)}} \cap A^{(0)}_{[0,T)})) \subset \text{convhull}(\overline{T_A(A^{(n)}_{[0,T)}}))$$

This then implies $T_A(A^{(n+1)}_{[0,T)}) \subset A^{(n+1)}_{[0,T)}$, as desired. Thus we need to prove

Theorem 5.1. *Under the assumptions of Theorem 4.5, we have $T_A(A^{(n)}_{[0,T)}) \subset A^{(0)}_{[0,T)}$.*

This shall occupy the rest of the paper

Proof. In order to prove this, we iterate the map T_A, i. e. we show that after applying sufficiently many iterations of T_A to a function in $A^{(0)}_{[0,T)}$, and assuming that each iterate up to but excluding the last sits in $A^{(0)}_{[0,T)}$, one again lands in A_0 for the last iterate. The proof will also easily imply that applying any number of iterations T^n_A to the trivial tuple $\Gamma_{trivial}$ takes one into $A^{(0)}_{[0,T)}$. Further, it will then also follow from the proof

[46] where the implied constant is independent of T

that the stronger assertion of the theorem holds, i. e. that we can alternate iteration of T_A with formation of convex linear combinations. Our procedure shall simply be to apply T_A to a given tuple $\Gamma = \{\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}, \ldots\}$, which is assumed to sit in $A^{(0)}$, and if necessary assume that $\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}$ itself is defined by the righthand side of (3.45) with respect to a different tuple Γ', and similarly for the other ingredients in Γ. In order to simplify notation, we shall not even distinguish between these tuples, i. e. *we proceed as for the derivation of a priori estimates.* We shall commence by retrieving the estimates (3.48) for the radiation part. However, before being able to do so, we need to justify the assertion made earlier about the phase $(\Psi - \Psi_\infty)_2(t, y)$:

Lemma 5.2. *Assume that the relations* (3.32) *are satisfied. Then we have for $\phi \in \mathcal{A}$ (recall the definitions after* (3.48)*)*

$$\phi(y)|(\Psi - \Psi_\infty)_2|(t, y) \lesssim \delta^2 \langle t\rangle^{-\frac{3}{2}+\delta_1}$$

Proof. Recall the definition

$$\begin{aligned}(\Psi - \Psi_\infty)_2(t, y) =& y[\omega(t)\nu(t) - \frac{\beta(t)}{2}\nu(t)\lambda(t)(\mu_\infty - \mu)(t) - \frac{a_\infty v_\infty - \frac{b_\infty y_\infty}{2}}{a_\infty + b_\infty t}] \\ &+ y^2[\frac{b_\infty}{4(a_\infty + b_\infty t)} - \frac{\beta}{4}\nu^2(t) - \frac{\beta}{4}\lambda^2(t)(\mu_\infty - \mu)^2(t)]\end{aligned}$$

The claimed estimate now follows easily from the facts that $c_\infty = a_\infty v_\infty - \frac{b_\infty y_\infty}{2}$ as well as $|\omega(t) - \frac{c_\infty}{\lambda_\infty(t)}| \lesssim \delta^2\langle t\rangle^{-\frac{3}{2}+\delta_1}$, $|\beta(t)\nu(t) - \frac{b_\infty}{\lambda_\infty(t)}| \lesssim \delta^2\langle t\rangle^{-\frac{3}{2}+\delta_1}$, $|\nu(t) - 1| \lesssim \delta^2\langle t\rangle^{-\frac{1}{2}+\delta_1}$. □

We also need

Lemma 5.3. *The following estimate holds under the same assumptions as in the preceding Lemma:*

$$|(\Psi - \Psi_\infty)_1(t)| \lesssim \delta^2\langle t\rangle^{\frac{1}{2}+\delta_1}$$

Proof. This is along the same lines, although the algebra is a bit more complicated. Observe that

$$\frac{d}{ds}[\frac{v_\infty^2 s a_\infty}{a_\infty + b_\infty s} - \frac{b_\infty v_\infty s y_\infty}{a_\infty + b_\infty s} + \gamma_\infty - \frac{b_\infty y_\infty^2}{4(a_\infty + b_\infty s)}] = \frac{(a_\infty v_\infty - \frac{b_\infty y_\infty}{2})^2}{(a_\infty + b_\infty s)^2} = \frac{c_\infty^2}{\lambda_\infty^2(s)},$$

while also $|\partial_s[\gamma_s - s] - c_\infty\lambda_\infty^{-2}(s)| \lesssim \delta^2 s^{-\frac{1}{2}+\delta_1}$. The claim follows easily from this and the definition of $(\Psi - \Psi_\infty)_1$. □

We shall first dispose of the easier estimates in (3.48), which happens to be everything with the exception of the *strong local dispersive estimate* and the *pseudo-conformal almost conservation law*, i. e. the third and fourth inequality. Commence with the case $k = 0$ and consider $C_0^{-1}\langle s\rangle^{\frac{1}{2}-\delta_2}||\tilde{U}_{dis}(s, y)||_{L_y^M}$. We shall employ the customary bootstrap technique. Thus we assume an estimate $C_0^{-1}\langle s\rangle^{\frac{1}{2}-\delta_2}||\tilde{U}_{dis}(s, y)||_{L_y^M} \leq \Lambda\delta$ for some sufficiently large[47] Λ, and similarly for all the other norms in (3.48) as well as the modulation parameters[48] etc., and then show that choosing $\delta > 0$ small enough implies the same inequalities with $\frac{\Lambda}{2}$ instead. Then using the local solvability and obvious continuous dependence of the norms on the time-interval we infer the desired a priori bound. Now use (3.45) as well as Duhamel's principle and Theorem 2.1 to deduce that

$$\begin{aligned}&C_0^{-1}\langle t\rangle^{\frac{1}{2}-\delta_2}||\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}(t, .)||_{L_x^M} \\ &\lesssim C_0^{-1}\langle t\rangle^{\frac{1}{2}-\delta_2}||e^{it\mathcal{H}}P_s[\begin{pmatrix} e^{-i(\Psi-\Psi_\infty)_1(t)} & 0 \\ 0 & e^{i(\Psi-\Psi_\infty)_1(t)} \end{pmatrix}[\begin{pmatrix} A(. + \lambda_\infty(\mu - \mu_\infty)(t)) \\ \bar{A}(. + \lambda_\infty(\mu - \mu_\infty)(t)) \end{pmatrix}_{dis} \\ &+ \sum_{j=1}^{6}\alpha_j\eta_{j,\text{proper}}(. + \lambda_\infty(\mu - \mu_\infty)(t))]]||_{L_x^M} + C_0^{-1}\langle t\rangle^{\frac{1}{2}-\delta_2}\int_0^t ||e^{i(t-s)\mathcal{H}}[...]_{dis}(s, .)||_{L_x^M} ds\end{aligned}$$

[47]Here the size depends on certain a priori constants independent of δ

[48]More precisely, one replaces δ by $\Lambda\delta$ and $\lesssim$ by $\leq$

The first two terms here are easy to estimate with respect to $||.||_{L_t^M}$ on account of (3.32) as well as Theorem 2.1, if one chooses Λ large enough in relation to $||A||_{L_x^1\cap L_x^2}$; one thus gets a bound of the form $\leq \frac{\Lambda}{100}\delta+\Lambda^2\delta^2$ upon choosing Λ large enough for the contribution of these terms, and this allows one to close if δ is chosen small enough in relation to Λ. Thus we now focus on the last integral term, in which we recall $[...]_{dis}$ stands for a sum of expressions, which basically fall into two contributions, namely local as well as non-local terms. Of the local ones, the most difficult contribution is easily seen[49] to come from the following term[50] in $[...]_{dis}$:

$$[\begin{pmatrix} 0 & -e^{2i(\Psi-\Psi_\infty)_1(s)-2i(\Psi-\Psi_\infty)_1(t)}+1 \\ e^{-2i(\Psi-\Psi_\infty)_1(s)+2i(\Psi-\Psi_\infty)_1(t)}-1 & 0 \end{pmatrix}\phi_0^4\begin{pmatrix} \tilde{U}^{(t)}(s) \\ \overline{\tilde{U}^{(t)}(s)} \end{pmatrix}]_{dis},$$

which leads to the expression

$$C_0^{-1}\langle t\rangle^{\frac{1}{2}-\delta_2}\int_0^t ||e^{i(t-s)\mathcal{H}}[\begin{pmatrix} 0 & -e^{2i(\Psi-\Psi_\infty)_1(s)-2i(\Psi-\Psi_\infty)_1(t)}+1 \\ e^{-2i(\Psi-\Psi_\infty)_1(s)+2i(\Psi-\Psi_\infty)_1(t)}-1 & 0 \end{pmatrix}$$
$$\phi_0^4\begin{pmatrix} \tilde{U}^{(t)}(s) \\ \overline{\tilde{U}^{(t)}(s)} \end{pmatrix}]_{dis}||_{L_x^M}ds$$

Observe from Lemma 5.3 that we have

$$\sup_{0\leq s\leq\delta^{-\frac{1}{2}}}|(\Psi-\Psi_\infty)_1|(s)\lesssim\Lambda\delta^{\frac{7}{4}-\delta_1},$$

whence we get if we restrict $t<\delta^{-\frac{1}{2}}$ and $M>>\delta_2^{-1}$

$$C_0^{-1}||\chi_{<\delta^{-\frac{1}{2}}}(t)\langle t\rangle^{\frac{1}{2}-\delta_2}\int_0^t ||e^{i(t-s)\mathcal{H}}[\begin{pmatrix} 0 & -e^{2i(\Psi-\Psi_\infty)_1(s)-2i(\Psi-\Psi_\infty)_1(t)}+1 \\ e^{-2i(\Psi-\Psi_\infty)_1(s)+2i(\Psi-\Psi_\infty)_1(t)}-1 & 0 \end{pmatrix}$$
$$\phi_0^4\begin{pmatrix} \tilde{U}^{(t)}(s) \\ \overline{\tilde{U}^{(t)}(s)} \end{pmatrix}]_{dis}||_{L_x^M}ds||_{L_t^M}$$
$$\lesssim C_0^{-1}\Lambda^2\delta^{\frac{11}{4}-\delta_1}||\langle t\rangle^{\frac{1}{2}-\delta_2}\int_0^t (t-s)^{-\frac{1}{2}+\frac{1}{M}}\langle s\rangle^{-\frac{3}{2}+\delta_3}ds||_{L_t^M}\lesssim C_0^{-1}\Lambda^2\delta^{\frac{11}{4}-\delta_1}$$

upon invoking (3.48), (3.47), which in turn can be bounded by $\leq\frac{\Lambda}{100}\delta$ upon choosing δ etc small enough. Now assume that $t\geq\delta^{-\frac{1}{2}}$. Then we get

$$C_0^{-1}||\chi_{>\delta^{-\frac{1}{2}}}(t)\langle t\rangle^{\frac{1}{2}-\delta_2}\int_0^t ||e^{i(t-s)\mathcal{H}}[\begin{pmatrix} 0 & -e^{2i(\Psi-\Psi_\infty)_1(s)-2i(\Psi-\Psi_\infty)_1(t)}+1 \\ e^{-2i(\Psi-\Psi_\infty)_1(s)+2i(\Psi-\Psi_\infty)_1(t)}-1 & 0 \end{pmatrix}$$
$$\phi_0^4\begin{pmatrix} \tilde{U}^{(t)}(s) \\ \overline{\tilde{U}^{(t)}(s)} \end{pmatrix}]_{dis}||_{L_x^M}ds||_{L_t^M}$$
$$\lesssim C_0^{-1}\Lambda\delta||\chi_{>\delta^{-\frac{1}{2}}}(t)\langle t\rangle^{\frac{1}{2}-\delta_2}\int_0^t (t-s)^{-\frac{1}{2}+\frac{1}{M}}\langle s\rangle^{-\frac{3}{2}+\delta_3}ds||_{L_t^M}\lesssim C_0^{-1}\delta^{-\frac{1}{M}+\frac{\delta_2}{2}}\delta\Lambda,$$

which is also $\leq\frac{\Lambda}{100}\delta$ upon choosing δ small enough. The remaining local terms in $[...]_{dis}$ can be handled analogously, so we now consider the contribution of the non-local term, which is

$$C_0^{-1}||\langle t\rangle^{\frac{1}{2}-\delta_2}||\int_0^t e^{i(t-s)\mathcal{H}}\begin{pmatrix} |\tilde{U}^{(t)}|^4\tilde{U}^{(t)}(s,.) \\ -|\tilde{U}^{(t)}|^4\overline{\tilde{U}^{(t)}(s,.)} \end{pmatrix}ds||_{L_x^M}||_{L_t^M}$$

Again referring to (3.48), as well as Theorem 2.1, and using Hölder's inequality, we can bound this by

$$\lesssim C_0^{-1}||\langle t\rangle^{\frac{1}{2}-\delta_2}(\Lambda C_0)^5\delta^5[\int_0^t[(t-s)^{-\frac{1}{2}+\frac{1}{M}}\langle s\rangle^{-\frac{3}{2}+3\delta_2}]^{\frac{M}{M-5}}ds]^{\frac{M-5}{M}}||_{L_t^M}\lesssim\frac{\Lambda}{100}\delta$$

[49]All the other local terms in (3.45) contain extra weights of at least the strength of $\nu-1$, and can be handled similarly.
[50]Recall the definition of $\tilde{U}^{(t)}(s,.)$ in (3.43).

upon choosing $\delta > 0$ small enough, as desired. The estimate for $||\langle t\rangle^{-\delta_2}||\tilde{U}(t,.)||_{L_x^2}||_{L_t^M}$ is carried out similarly and omitted. Moreover, we postpone retrieving the difficult strong local dispersive estimate, i. e. the global bound for $||\langle s\rangle^{\frac{3}{2}-\delta_3}\phi\tilde{U}||_{L_s^\infty L_x^\infty}$, as well as the pseudo-conformal almost conservation law, until later. This then completes the case $k=0$.
We move on to the case $k=1$. We start by retrieving control over

$$\sup_{\phi\in\mathcal{A}}||\langle s\rangle^{1-20\delta_2}||\phi\partial_x\tilde{U}_{dis}(s,x)||_{L_x^M}||_{L_s^M},\ ||\langle s\rangle^{-10\delta_2}||\partial_x\tilde{U}_{dis}(s,x)||_{L_x^2}||_{L_s^M},$$

which we do in tandem. Note that the operation ∂_x here occurs after projecting onto the dispersive part. We use the decomposition

$$\partial_x[\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis}]=[\partial_x[\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis}]]_{dis}-\sum_{i=1}^{6}\langle[\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis},\partial_x\xi_{k(i),\text{proper}}\rangle\eta_{i,\text{proper}}$$

The 2nd term here is then estimated using what we know from case $k=0$, whence we must estimate the first term, which we write as $\left(\begin{array}{c}(\partial_x\tilde{U}_{dis})_{dis}\\ \overline{(\partial_x\tilde{U}_{dis})_{dis}}\end{array}\right)$. Assuming that the quantity

$$B(t):=C_1^{-1}[\sup_{\phi\in\mathcal{A}}||\langle s\rangle^{1-20\delta_2}||\phi(\partial_x\tilde{U}_{dis})_{dis}(s,.)||_{L_x^M}||_{L_s^M([0,t])}+||\langle s\rangle^{-10\delta_2}||(\partial_x\tilde{U}_{dis})_{dis}(s,.)||_{L_x^2}||_{L_s^M([0,t])}]\leq\Lambda\delta,$$

we shall boost this to the bound $\lesssim\frac{\Lambda}{100}\delta$. Note that the assumption also gives us the corresponding bounds for $\partial_x U$ upon using the already established estimates in the case $k=0$. Commence with the expression

$$C_1^{-1}\sup_{\phi\in\mathcal{A}}||\langle s\rangle^{1-20\delta_2}||\phi(\partial_x\tilde{U}_{dis})_{dis}(s,x)||_{L_x^M}||_{L_s^M([0,t])}$$

If we project (1.8) onto its dispersive part, differentiate with respect to x, and then project again onto the dispersive part, we produce additional local source terms of the schematic forms VU, $\langle V,U\rangle W$ for certain Schwartz functions V,W, in addition to terms involving $\partial_x U$. The former terms are handled by using the already established estimates for $k=0$ as well as the assumption $C_1>>C_0$. As for the latter, the most difficult local term leads to the Duhamel term[51]

$$C_1^{-1}||\langle s\rangle^{1-20\delta_2}\int_0^s||\phi(x)e^{i(s-\lambda)\mathcal{H}}[\left(\begin{array}{cc}0 & -e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)}+1\\ e^{-2i(\Psi-\Psi_\infty)_1(\lambda)+2i(\Psi-\Psi_\infty)_1(s)}-1 & 0\end{array}\right)\phi_0^4\left(\begin{array}{c}\partial_x\tilde{U}^{(s)}(\lambda)\\ \partial_x\bar{\tilde{U}}^{(s)}(\lambda)\end{array}\right)]_{dis}||_{L_x^M}d\lambda||_{L_s^M([0,t])}$$

First restrict integration to the interval $[0,s-\delta^{-\frac{1}{2}}\langle s\rangle^{\frac{6}{M}}]$. This we can estimate crudely by[52]

$$\lesssim\Lambda\delta C_1^{-1}||\langle s\rangle^{1-20\delta_2+\frac{1}{M}}\int_0^{s-\delta^{-\frac{1}{2}}\langle s\rangle^{\frac{6}{M}}}\langle s\rangle^{\frac{2}{M}}\langle s-\lambda\rangle^{-\frac{3}{2}}\langle\lambda\rangle^{-1+20\delta_2}d\lambda||_{L_s^M([0,t])}\lesssim\frac{\Lambda}{100}\delta$$

upon choosing δ small enough. On the interval $[s-\delta^{-\frac{1}{2}}\langle s\rangle^{\frac{6}{M}},s]$, one proceeds similarly, but exploits the fact that thanks to the proof of Lemma 5.3, we have

$$\sup_{\lambda\in[s-\delta^{-\frac{1}{2}}\langle s\rangle^{\frac{6}{M}},s]}|-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)}+1|\lesssim\delta^{\frac{3}{2}}\langle s\rangle^{-\frac{1}{2}+\delta_1+\frac{6}{M}}$$

[51]One uses the auxiliary variable $\partial_x\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis}^{(t)}(s,x):=\left(\begin{array}{c}\partial_x\tilde{U}_{dis}^{(t)}(s,x)\\ \partial_x\bar{\tilde{U}}_{dis}^{(t)}(s,x)\end{array}\right):=\left(\begin{array}{c}e^{-i(\Psi-\Psi_\infty)_1(t)}\partial_x U_{dis}(s,x+\lambda_\infty(\mu-\mu_\infty)(t))\\ e^{i(\Psi-\Psi_\infty)_1(t)}\overline{\partial_x U_{dis}(s,x+\lambda_\infty(\mu-\mu_\infty)(t))}\end{array}\right)$, analogously to $\left(\begin{array}{c}\tilde{U}^{(t)}(s,.)\\ \bar{\tilde{U}}^{(t)}(s,.)\end{array}\right)$ which we used before. We similarly introduce $\partial_x\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)^{(t)}:=\left(\begin{array}{c}\partial_x\tilde{U}^{(t)}\\ \partial_x\bar{\tilde{U}}^{(t)}\end{array}\right)$.

[52]One again uses Hölder's inequality to handle the fact that we only control $||\phi\partial_x\tilde{U}^{(s)}(\lambda,x)||_{L_\lambda^M L_x^M}$.

The remaining local terms involving $\partial_x\tilde{U}^{(t)}$ are easier and estimated similarly. Next, consider the non-local term. We have to estimate

$$C_1^{-1}||\langle s\rangle^{1-20\delta_2}||\phi\int_0^s e^{i(s-\lambda)\mathcal{H}}\left(\begin{array}{c}|\tilde{U}^{(s)}(\lambda,.)|^4\overline{\partial_x\tilde{U}^{(s)}}(\lambda,.)\\ -|\tilde{U}^{(s)}(\lambda,.)|^4\overline{\partial_x\tilde{U}^{(s)}}(\lambda,.)\end{array}\right)_{dis}d\lambda||_{L_x^M}||_{L_s^M([0,t])}$$

in addition to similar terms (Leibnitz rule). If we invoke the assumption on $B(t)$, the weighted estimates in Theorem 2.1 as well as the estimates for the case $k=0$ we can bound this by

$$\lesssim \Lambda^5C_0^4\delta^5||\langle s\rangle^{1-20\delta_2}\int_0^s(s-\lambda)^{-1+20\delta_2-\frac{1}{M}}\langle\lambda\rangle^{4\delta_2+10\delta_2-\frac{3}{2}}\langle\lambda\rangle^{\frac{1}{2}-20\delta_2+\frac{6}{M}}d\lambda||_{L_s^M([0,t])}\lesssim\frac{\Lambda}{100}\delta$$

We move on to control $||\langle s\rangle^{-10\delta_2}||(\partial_x\tilde{U}_{dis})_{dis}(s,x)||_{L_x^2}||_{L_s^M([0,t])}$. Note that *we cannot simply reiterate the Duhamel procedure here, since this would lead to a loss for the local terms*[53]. Observe that (1.8) implies the following schematic relation:

$$\begin{aligned}(5.1)\qquad & i\partial_s[\partial_x\tilde{U}^{(t)}_{dis}](s,.)+\partial_x^2[\partial_x\tilde{U}^{(t)}_{dis}](s,.)-\partial_x\tilde{U}^{(t)}_{dis}(s,.)\\ &=[V\partial_x\tilde{U}^{(t)}](s,.)+[V\overline{\partial_x\tilde{U}^{(t)}}](s,.)+\ldots+[|\tilde{U}^{(t)}|^4\partial_x\tilde{U}^{(t)}](s,.),\end{aligned}$$

where the subscripts really refer to the top entry of the dispersive part of the corresponding vector-valued function. As usual V refers to certain Schwartz functions which may depend on t and s. We deduce that

$$\begin{aligned}(5.2)\qquad & i\partial_\lambda\int_{-\infty}^{\infty}|\partial_x\tilde{U}^{(s)}_{dis}|^2(\lambda,.)dx=2\Im\int_{-\infty}^{\infty}i\partial_\lambda[e^{i\lambda}\partial_x\tilde{U}^{(s)}_{dis}](\lambda,.)\overline{e^{i\lambda}\partial_x\tilde{U}^{(s)}_{dis}(\lambda,.)}dx\\ &=2\Im\int_{-\infty}^{\infty}[V\partial_x\tilde{U}^{(s)}_{dis}(\lambda,.)\overline{\partial_x\tilde{U}^{(s)}_{dis}(\lambda,.)}dx+\ldots+|\tilde{U}^{(s)}|^4\partial_x\tilde{U}^{(s)}\overline{\partial_x\tilde{U}^{(s)}}(\lambda,.)dx\end{aligned}$$

Using the already improved local bound, we can estimate

$$||\langle s\rangle^{-20\delta_2}\int_0^s|\Im\int_{-\infty}^{\infty}[V\partial_x\tilde{U}^{(s)}(\lambda,.)\overline{\partial_x\tilde{U}^{(s)}(\lambda,.)}dx|d\lambda||_{L_s^M([0,t])}\lesssim(\frac{C_1\Lambda}{100})^2\delta^2\int_0^s\langle\lambda\rangle^{-2(1-20\delta_2)}d\lambda,$$

which is integrable in λ upon choosing δ_2 small enough. The remaining expressions on the righthand side of (5.2) can be estimated similarly, combining the preceding estimates we get the improved bound on $B(t)$, as desired.

Next, consider the norm $||\langle s\rangle^{\frac{1}{2}-25^k\delta_2}||(\partial_x\tilde{U}_{dis})_{dis}(s,.)||_{L_x^M}||_{L_s^M([0,T))}$. This we estimate by reverting to the usual Duhamel's formula; we treat here the most difficult local term (schematically)

$$[-e^{-2i(\Psi-\Psi_\infty)_1(\lambda)+2i(\Psi-\Psi_\infty)_1(s)}+1]V\partial_xU(\lambda,.)$$

the others following in a similar vein[54]: we have

$$\begin{aligned}&||\langle s\rangle^{\frac{1}{2}-25\delta_2}||\int_0^s e^{i(s-\lambda)\mathcal{H}}[(-e^{-2i(\Psi-\Psi_\infty)_1(\lambda)+2i(\Psi-\Psi_\infty)_1(s)}+1)V\partial_xU]_{dis}(\lambda,.)d\lambda||_{L_x^M}||_{L_s^M}\\ &\lesssim\frac{\Lambda C_1}{100}\delta||\langle s\rangle^{\frac{1}{2}-25\delta_2}\int_0^s(s-\lambda)^{-\frac{1}{2}+\frac{1}{M}}\lambda^{-1+20\delta_2}d\lambda||_{L_s^M}\end{aligned}$$

This is easily seen to also improve the bound $\Lambda\delta C_1$, if necessary by improving the preceding estimates for $B(t)$. In order to complete the case $k=1$, we still need to retrieve control over $K^{-1}\sup_{t\in[0,T]}||C\partial_x\tilde{U}_{dis}||_{L_x^2}$, which we shall do later.

Proceeding to higher derivatives $k\geq 2$ is an elementary induction, recycling the same estimates. Observe that if one differentiates the quintilinear non-local term k times, one obtains schematically either $|\tilde{U}^{(s)}|^4\partial_x^k(\tilde{U}^{(s)})$,

[53]More precisely, it appears that the extra factor $\langle s\rangle^{-10\delta_2}$ should allow one to gain a bit; however, the loss from the weak local decay control, i. e. $||\phi\partial_xU(s,.)||_{L_x^\infty}\lesssim\langle s\rangle^{-1+20\delta_2}$, is too much. On the other hand, if one strengthened the weight in the L^2-norm to $\langle s\rangle^{-(20+)\delta_2}$, one would encounter difficulties in retrieving the weak local estimate. The reader may ask why we don't build in the strong local decay for all higher derivatives to begin with. The problem with this is that we would have to gain control over more and more derivatives that way, indeed forcing control over infinitely many derivatives.

[54]Indeed, one gains extra weights like $\nu-1$ for the other local terms, and can proceed analogously. For the non-local term, use that $|||\tilde{U}^{(s)}|^4\partial_x\tilde{U}^{(s)}(\lambda,.)||_{L_x^{M'}}\lesssim\langle\lambda\rangle^{-\frac{3}{2}+4\delta_2}||\partial_x\tilde{U}^{(s)}(\lambda,.)||_{L_x^2}$.

or else $\partial_x^{k-1}(\tilde{U}^{(s)})\partial_x(\tilde{U}^{(s)})(\tilde{U}^{(s)})^2\overline{\tilde{U}^{(s)}}$ or else terms of the form $\partial_x^{\alpha_1}(\tilde{U}^{(s)})\partial_x^{\alpha_2}(\overline{\tilde{U}^{(s)}})\dots\partial_x^{\alpha_5}(\tilde{U}^{(s)})$, where all $\alpha_i < k-1$, or terms equivalent to these for all intents and purposes. The first term in this list one can treat just as before, using the estimates for U. For the 2nd, one estimates

$$||(\partial_x^{k-1}(\tilde{U}^{(s)})\partial_x(\tilde{U}^{(s)})(\tilde{U}^{(s)})^2\overline{\tilde{U}^{(s)}})(\lambda,.)||_{L_x^{M'}} \leq \Lambda^5 C_{k-1}C_1C_0^3\delta^5\lambda^{10^{k-1}\delta_2+10\delta_2+3\delta_2}\langle\lambda\rangle^{-\frac{3}{2}}$$
$$\times ||\langle\lambda\rangle^{\frac{1}{2}-\delta_2}\tilde{U}^{(s)}(\lambda,.)||_{L_x^M}||\langle\lambda\rangle^{-10^{k-1}\delta_2}\partial_x^{k-1}\tilde{U}^{(s)}(\lambda,.)||_{L_x^{2+}}||\langle\lambda\rangle^{-\delta_2}\partial_x\tilde{U}^{(s)}(\lambda,.)||_{L_x^2},$$

and we have

$$\lambda^{10^{k-1}\delta_2+10\delta_2+3\delta_2}\langle\lambda\rangle^{-\frac{3}{2}} \lesssim \langle\lambda\rangle^{-\frac{3}{2}+15^k\delta_2}$$

Thus one can comfortably absorb an extra weight $\lambda^{\frac{1}{2}-20^k\delta_2+\frac{4}{M}}$ here, and continues as before. The estimate for the last term in the above list is similar. Time derivatives can now be handled upon turning them into spatial derivatives via (1.8).

Finally, one retrieves control over

$$\sum_{1\leq k\leq N-1} K^{-k}\sup_{3i+j=k}||C\partial_s^i\partial_y^j\tilde{U}_{dis}(s,.)||_{L_s^M L_y^2([0,T])}]$$

by putting $K = \langle T\rangle^{100}$, say, using the fact that $C = x-2pt$ and $i\partial_t + \triangle$ commute, and using the already improved estimates as well as crude bounds. We also observe that the preceding estimates can easily be bootstrapped to yield the final bounds in Theorem 4.5.

We now have to come to terms with the *strong local dispersive estimate*, or SLDE, as well as the *pseudo-conformal almost conservation*, i. e. the expressions

$$\sup_{\phi\in\mathcal{A}}\sup_{t\geq 0}\langle t\rangle^{\frac{3}{2}-\delta_3}||\phi\tilde{U}_{dis}(s,.)||_{L_x^\infty}, \sup_{t\geq 0}||C\tilde{U}_{dis}(t,.)||_{L_x^2}$$

We commence with SLDE in the following subsection.

5.1. Retrieving the strong local dispersion. [55]

We now need to show that we can deduce the inequality $\sup_{\phi\in\mathcal{A}}\sup_{0\leq t}\langle t\rangle^{\frac{3}{2}-\delta_3}||\phi\tilde{U}(t,.)||_{L_x^\infty} \leq \frac{\Lambda}{100}\delta$. For this we need to employ (3.45), which forces us to distinguish between the different kinds of expressions on the right hand side. Clearly we may assume $t\geq 1$. We first observe that Theorem 2.1 in conjunction with our assumptions on $A(.)$ as well as (3.32), (3.47), (3.48) and Lemma 5.3 imply that the free contribution is acceptable, with a t^{δ_3} to spare:

$$\sup_{0\leq t}\langle t\rangle^{\frac{3}{2}}||e^{it\mathcal{H}}P_s[\begin{pmatrix} e^{i(\Psi_\infty-\Psi)_1(t)} & 0 \\ 0 & e^{-i(\Psi_\infty-\Psi)_1(t)}\end{pmatrix}[\begin{pmatrix} A(.+\lambda_\infty(\mu-\mu_\infty)(t)) \\ \bar{A}(.+\lambda_\infty(\mu-\mu_\infty)(t))\end{pmatrix}_{dis}$$
$$+\sum_{j=1}^6\alpha_j\eta_{j,\mathrm{proper}}(.+\lambda_\infty(\mu-\mu_\infty)(t))]]|| \leq \frac{\Lambda}{100}\delta,$$

upon choosing $\Lambda > 0$ large enough. We next subdivide $[...]_{dis}$ in the integrand of the Duhamel term in (3.45) into local and non-local contributions. As for the local contributions, as usual the most difficult is (schematically)

$$\int_0^t e^{i(t-s)\mathcal{H}}[-e^{-2i(\Psi-\Psi_\infty)_1(s)+2i(\Psi-\Psi_\infty)_1(t)}+1][V\tilde{U}^{(t)}]_{dis}(s,.)ds$$

One estimates (using Theorem 2.1)

$$\langle t\rangle^{\frac{3}{2}-\delta_3}||\phi(x)\int_0^{t-\delta^{-\frac{1}{2}}} e^{i(t-s)\mathcal{H}}[-e^{-2i(\Psi-\Psi_\infty)_1(s)+2i(\Psi-\Psi_\infty)_1(t)}+1][V\tilde{U}^{(t)}]_{dis}(s,.)ds||_{L_x^\infty}$$
$$\lesssim \langle t\rangle^{\frac{3}{2}-\delta_3}\Lambda\delta\int_0^{t-\delta^{-\frac{1}{2}}}\langle t-s\rangle^{-\frac{3}{2}}\langle s\rangle^{-\frac{3}{2}+\delta_3}ds \lesssim \Lambda\delta^{1+\frac{\delta_3}{2}},$$

[55] The argument to follow is certainly not optimal; however, it allows us to gain a refined understanding which will play an important role for the bilinear estimates needed to control λ_6. A large simplification would result if one could improve the SLDE to not contain any losses.

which leads to the bound $\lesssim \frac{\Lambda}{100}\delta$ upon choosing $\delta > 0$ small enough. Moreover, using Lemma 5.3, one gets the same estimate for the integral over $[t-\delta^{-\frac{1}{2}}, t]$, as desired. The remaining local terms in $[...]_{dis}$ can be handled similarly, whence we now turn to the real task, dealing with the non-local term, i. e. the expression

$$\langle t\rangle^{\frac{3}{2}-\delta_3}||\phi(x)\int_0^t e^{i(t-s)\mathcal{H}}\left(\begin{array}{c}|\tilde{U}^{(t)}|^4\tilde{U}^{(t)}(s,.)\\ -|\tilde{U}^{(t)}|^4\overline{\tilde{U}^{(t)}}(s,.)\end{array}\right)_{dis}ds||_{L_x^\infty}$$

We intend to turn this into an expression of the following form:

$$\langle t\rangle^{\frac{3}{2}-\delta_3}\langle\int_0^t e^{i(t-s)\mathcal{H}}\left(\begin{array}{c}|\tilde{U}^{(t)}|^4\tilde{U}^{(t)}(s,.)\\ -|\tilde{U}^{(t)}|^4\overline{\tilde{U}^{(t)}}(s,.)\end{array}\right)_{dis}ds, \left(\begin{array}{c}\phi\\ \psi\end{array}\right)\rangle,$$

for suitable Schwartz functions ϕ, ψ. The device for achieving this is the discrete Fourier transform. First, using a partition of unity $\{\phi_i\}$ subordinate to intervals of length 2π, we reduce to estimating

$$\phi_j(x)\phi(x)\int_0^t e^{i(t-s)\mathcal{H}}\left(\begin{array}{c}|\tilde{U}^{(t)}|^4\tilde{U}^{(t)}(s,.)\\ -|\tilde{U}^{(t)}|^4\overline{\tilde{U}^{(t)}}(s,.)\end{array}\right)_{dis}ds$$

Write (we omit the superscripts $\tilde{}$ and ${}^{(t)}$ from now on as they are irrelevant in this argument)

$$i\phi\phi_j\int_0^t e^{i(t-s)\mathcal{H}}\left(\begin{array}{c}|U|^4U(s)\\ -|U|^4\bar{U}(s)\end{array}\right)_{dis}ds = \sum_{n\in\mathbf{Z}}\left(\begin{array}{c}a_{nj}e^{in(x-x_j)}\\ \overline{a_{nj}}e^{-in(x-x_j)}\end{array}\right)$$

We have

$$\Re a_{nj} = \frac{i}{4\pi}\langle\phi\phi_j\int_0^t e^{i(t-s)\mathcal{H}}\left(\begin{array}{c}|U|^4U(s)\\ -|U|^4\bar{U}(s)\end{array}\right)_{dis}ds, \left(\begin{array}{c}e^{in(x-x_j)}\\ e^{-in(x-x_j)}\end{array}\right)\rangle,$$

$$\Im a_{nj} = \frac{1}{4\pi}\langle\phi\phi_j\int_0^t e^{i(t-s)\mathcal{H}}\left(\begin{array}{c}|U|^4U(s)\\ -|U|^4\bar{U}(s)\end{array}\right)_{dis}ds, \left(\begin{array}{c}e^{in(x-x_j)}\\ -e^{-in(x-x_j)}\end{array}\right)\rangle$$

Thus for example

$$\Re a_{nj} = \frac{i}{4\pi}\langle\int_0^t e^{i(t-s)\mathcal{H}}\left(\begin{array}{c}|U|^4U(s)\\ -|U|^4\bar{U}(s)\end{array}\right)_{dis}ds, \left(\begin{array}{c}\phi\phi_j e^{in(x-x_j)}\\ \phi\phi_j e^{-in(x-x_j)}\end{array}\right)\rangle,$$

and similarly for the imaginary part. In order to be able to carry out the summation over n, we need to carry out an integration by parts. Another way to approach this is to note that only moderately small values of n, i. e. $|n| < t^\epsilon$, contribute since we can independently control $||\partial_x^N U||_{L_x^2}$, whence the large frequency part of U can be made arbitrarily small. Indeed, note that we have

$$\langle\left(\begin{array}{c}U\\ \bar{U}\end{array}\right), \phi e^{inx}\rangle = \frac{1}{(in)^N}\langle\partial_x^N[\phi\left(\begin{array}{c}U\\ \bar{U}\end{array}\right)], e^{inx}\rangle$$

Thus we may and shall assume that $|n| < \delta^{-\epsilon}t^\epsilon$, for $\epsilon > \epsilon_0(N) > 0$, N as in (3.48), such that $\lim_{N\to\infty}\epsilon_0(N) = 0$. We thus lose $\delta^{-\epsilon}\langle t\rangle^\epsilon$ in the end, which we can afford since we may arrange $\delta_3 >> \epsilon_0(N)$. We now need to estimate the following expression:

$$\int_0^t\langle\left(\begin{array}{c}|U|^4U(s)\\ -|U|^4\bar{U}(s)\end{array}\right), e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}\rangle ds$$

We distinguish between the cases $s > \frac{t}{2}$, $s \le \frac{t}{2}$, which aside from a simple technicality are treated by the same method. We shall use the notation $P_{\le a}$, P_a, $P_{>a}$, $a \in \mathbf{R}_{>0}$ dyadic[56], for the standard Littlewood-Paley multipliers, see e. g. [St]. We shall also assume that $\tilde{U}$ satisfies pointwise-in-time estimates below in order to simplify the exposition; thus we shall assume bounds of the form $\langle s\rangle^{\frac{1}{2}-\delta_2}||\tilde{U}(s,.)||_{L_x^\infty} \le \delta\Lambda$ etc. It will be straightforward to adjust the arguments below to the case of weighted-in-time norms, since we assume $M >> \delta_2^{-1}$ and hence we can absorb losses of order $\langle s\rangle^{O(\frac{1}{M})}$. Finally, as it is clear that we gain lots of δ's below, we shall occasionally omit the Λ.

[56] We shall more generally mean P_α for $\alpha \in \mathbf{R}_{>0}$ to denote P_j if $2^{j-1} \le \alpha < 2^j$

Case A: $s < \frac{t}{2}$. The idea is to exploit the pseudo-conformal operator to reduce at least one of the two $|U|^2$'s to frequency $< s^{-\frac{3}{4}}$. In this case, one exploits the fact that the distorted Fourier transform vanishes at the origin. We start by chopping things apart: specialize to the following two terms:

$$\int_0^{\frac{t}{2}} \langle \left(\begin{array}{c} \chi_{>0}|U|^4 U(s) \\ -\chi_{>0}|U|^4 \bar{U}(s) \end{array} \right), e^{-i(t-s)\mathcal{H}^*} \left(\begin{array}{c} \phi \\ \psi \end{array} \right)_{dis} \rangle ds$$

$$\int_0^{\frac{t}{2}} \langle \left(\begin{array}{c} \chi_{<0}|U|^4 U(s) \\ -\chi_{<0}|U|^4 \bar{U}(s) \end{array} \right), e^{-i(t-s)\mathcal{H}^*} \left(\begin{array}{c} \phi \\ \psi \end{array} \right)_{dis} \rangle ds,$$

where $\chi_{>0}$ is the Heaviside function localizing to $x > 0$. Both are treated the same way, so consider the first expression: rewrite it as

$$\int_0^{\frac{t}{2}} \langle \left(\begin{array}{c} \chi_{>0}|U|^4(s) \\ -\chi_{>0}|U|^4(s) \end{array} \right), \left(\begin{array}{c} \bar{U}(s) \\ U(s) \end{array} \right) \times e^{-i(t-s)\mathcal{H}^*} \left(\begin{array}{c} \phi \\ \psi \end{array} \right)_{dis} \rangle ds,$$

where $\times$ denotes componentwise multiplication[57]. We commence by reducing $\left(\begin{array}{c} \chi_{>0}|U|^4(s) \\ -\chi_{>0}|U|^4(s) \end{array} \right)$ to its dispersive part. To achieve this, note that

$$\left(\begin{array}{c} \chi_{>0}|U|^4(s) \\ -\chi_{>0}|U|^4(s) \end{array} \right) = \left(\begin{array}{c} \chi_{>0}|U|^4(s) \\ -\chi_{>0}|U|^4(s) \end{array} \right)_{dis} + \sum_{j=1}^{6} a_{k(j)} \langle \left(\begin{array}{c} \chi_{>0}|U|^4(s) \\ -\chi_{>0}|U|^4(s) \end{array} \right), \xi_{k(j),\mathrm{proper}} \rangle \eta_{j,\mathrm{proper}}$$

where $\xi_{j,\mathrm{proper}}$ is the basis for the generalized root space of $\mathcal{H}^*$, while $\eta_{j,\mathrm{proper}}$ is the basis for the generalized root space of $\mathcal{H}$, as explained earlier. The $a_{k(j)}$ are suitable numerical coefficients. Then observe that by the improved local dispersive estimate, we have ($\epsilon = \epsilon(\delta_3)$)

$$|\langle \left(\begin{array}{c} \chi_{>0}|U|^4(s) \\ -\chi_{>0}|U|^4(s) \end{array} \right), \xi_{j,\mathrm{proper}} \rangle| \lesssim (\Lambda C_0)^4 \delta^4 \langle s \rangle^{-6+\epsilon},$$

whence we treat the contribution of this part to the above integral expression by

$$\lesssim (\Lambda C_0)^4 \delta^4 \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{2}} \langle s \rangle^{-6+\epsilon} ds,$$

which is better than what we need. Thus we now consider

$$\int_0^{\frac{t}{2}} \langle \left(\begin{array}{c} \chi_{>0}|U|^4(s) \\ -\chi_{>0}|U|^4(s) \end{array} \right)_{dis}, \left(\begin{array}{c} \bar{U}(s) \\ U(s) \end{array} \right) \times e^{-i(t-s)\mathcal{H}^*} \left(\begin{array}{c} \phi \\ \psi \end{array} \right)_{dis} \rangle ds$$

Using the distorted Plancherel's Theorem 2.3 we can equate this with

$$\sum_{\pm} \int_{-\infty}^{\infty} \int_0^{\frac{t}{2}} \mathcal{F}_{\pm} \left(\begin{array}{c} \chi_{>0}|U|^4(s) \\ -\chi_{>0}|U|^4(s) \end{array} \right)(\xi) \overline{\tilde{\mathcal{F}}_{\pm} \left(\begin{array}{c} \bar{U}(s) \\ U(s) \end{array} \right) \times e^{-i(t-s)\mathcal{H}^*} \left(\begin{array}{c} \phi \\ \psi \end{array} \right)}(\xi) ds d\xi$$

Observe that we have $\mathcal{F}_{\pm}(\phi)(\xi) = \langle \phi, \sigma_3 e_{\pm}(x,\xi) \rangle$, $\tilde{\mathcal{F}}(\phi)(\xi) = \langle \phi, e_{\pm}(x,\xi) \rangle$. The cases $\pm$ are treated exactly analogously, so we stick with the $+$ case. Break the ξ-integral into two parts, one over $[0,\infty)$, the other over $(-\infty, 0]$. Commence with the case $\xi \in [0,\infty)$. Write

$$\int_0^{\infty} \int_0^{\frac{t}{2}} \mathcal{F} \left(\begin{array}{c} \chi_{>0}|U|^4(s) \\ -\chi_{>0}|U|^4(s) \end{array} \right)(\xi) \overline{\tilde{\mathcal{F}}[\chi_{>0} \left(\begin{array}{c} \bar{U}(s) \\ U(s) \end{array} \right) \times e^{-i(t-s)\mathcal{H}^*} \left(\begin{array}{c} \phi \\ \psi \end{array} \right)_{dis}]}(\xi) ds d\xi$$

$$= \int_0^{\infty} \int_0^{\frac{t}{2}} \langle \left(\begin{array}{c} \chi_{>0}|U|^4(s) \\ -\chi_{>0}|U|^4(s) \end{array} \right), s(\xi) e^{ix\xi} \underline{e} + \phi(x,\xi) \rangle \overline{\tilde{\mathcal{F}}[\chi_{>0} \left(\begin{array}{c} \bar{U}(s) \\ U(s) \end{array} \right) \times e^{-i(t-s)\mathcal{H}^*} \left(\begin{array}{c} \phi \\ \psi \end{array} \right)_{dis}]}(\xi) ds d\xi,$$

recalling Theorem 2.2. We first treat the simple contribution coming from the rapidly decaying function $\phi(x,\xi)$. As before, observe that

$$|\langle \left(\begin{array}{c} \chi_{>0}|U|^4(s) \\ \chi_{>0}|U|^4(s) \end{array} \right), \phi(x,\xi) \rangle| \lesssim \langle s \rangle^{-6+\epsilon},\ \epsilon = \epsilon(\delta_3)$$

[57] Also observe that we use the subscript $_{dis}$ both with reference to $\mathcal{H}$ as well as $\mathcal{H}^*$.

Indeed, we can estimate the L^2_ξ-norm of the function on the left in this fashion. Moreover, we have

$$||\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}](\xi)||_{L^2_\xi}\lesssim\langle t-s\rangle^{-\frac{3}{2}}s^{1+\epsilon(\delta_2)}$$

We are using here the pseudo-conformal almost conservation, which is part of our assumptions (3.48):

$$||(x-2sp)U(s,.)||_{L^2_x}\lesssim O(\delta),\, p=-i\frac{\partial}{\partial x}$$

Thus feeding in the preceding two estimates easily implies the desired bound. Now consider the difficult oscillatory part. Decompose

$$\int_0^\infty\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}\chi_{>0}|U|^4(s)\\ -\chi_{>0}|U|^4(s)\end{array}\right), s(\xi)e^{ix\xi}\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}}](\xi)dsd\xi=$$
$$\int_0^\infty\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}P_{\geq a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U|^2]\\ -P_{\geq a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U^2|]\end{array}\right), s(\xi)e^{ix\xi}\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}}](\xi)dsd\xi$$
$$+\int_0^\infty\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}P_{\geq a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2]\\ -P_{\geq a}[\chi_{>0}|U|^2(s)]P_{<a}[|U^2|]\end{array}\right), s(\xi)e^{ix\xi}\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}}](\xi)dsd\xi$$
$$+\int_0^\infty\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}P_{<a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U|^2]\\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U^2|]\end{array}\right), s(\xi)e^{ix\xi}\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}}](\xi)dsd\xi$$
$$+\int_0^\infty\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2]\\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U^2|]\end{array}\right), s(\xi)e^{ix\xi}\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}}](\xi)dsd\xi$$

The cutoff a here will be later chosen to be $\langle s\rangle^{-\frac{3}{4}}$. We treat each of the above terms separately. Start with the first, the **high-high case**: note that we can write

$$\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}](\xi)=\langle\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}, s(\xi)e^{ix\xi}\underline{e}+\phi(x,\xi)\rangle$$

Thus we can estimate

$$|\int_0^\infty\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}P_{\geq a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U|^2]\\ -P_{\geq a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U^2|]\end{array}\right), s(\xi)e^{ix\xi}\underline{e}\rangle\overline{\langle\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}, \phi(x,\xi)\rangle}dsd\xi|$$
$$\lesssim\int_0^{\frac{t}{2}}\langle s\rangle^{-3+\epsilon(\delta_3)}\langle t-s\rangle^{-\frac{3}{2}}ds\lesssim\langle t\rangle^{-\frac{3}{2}}$$

Hence we reduce to estimating the expression

$$\int_0^\infty\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}P_{\geq a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U|^2]\\ -P_{\geq a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U^2|]\end{array}\right), s(\xi)e^{ix\xi}\underline{e}\rangle\overline{\langle\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}, s(\xi)e^{ix\xi}\underline{e}\rangle}d\xi$$

We intend to use the ordinary Plancherel's Theorem here. We break this integral into two by including a multiplier $\phi_{(t^{-1000},t^{1000})}(\xi)$ or $\chi_{>0}(\xi)-\phi_{(t^{-1000},t^{1000})}(\xi)$, where $\phi_{(t^{-1000},t^{1000})}(\xi)$ smoothly localizes to the interval (t^{-1000},t^{1000}). It is easily seen that contribution obtained upon including the latter is very small (bounded by $\langle t\rangle^{-500}$), whence we may focus on the contribution of the former. By choosing $\phi_{(t^{-1000},t^{1000})}(\xi)$ suitably, we may assume that its Fourier transform has L^1-mass bounded by $\log t$. Now denote the Fourier multiplier with symbol $s(\xi)^2\phi_{(t^{-1000},t^{1000})}(\xi)$ by $\Pi_{(t^{1000},t^{1000})}$. Using ordinary Plancherel, we now reduce to estimating

$$\langle P_{\geq a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U|^2],\Pi_{(t^{-1000},t^{1000})}\langle\underline{e},\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}\rangle$$

We claim that we have

$$|\Pi_{(t^{-1000},t^{1000})}\langle\underline{e},\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}|\lesssim\langle t-s\rangle^{-\frac{3}{2}}\langle s\rangle^{\frac{1}{2}+\epsilon(\delta_2)}$$

This follows from $||xU(s)||_{L^\infty_x} \lesssim \langle s\rangle^{\frac{1}{2}+\epsilon(\delta_2)}$, which in turn is a consequence of

$$\begin{aligned}||xU||_{L^\infty_x} &\leq ||(x+2is\partial_x)U||_{L^\infty_x} + ||2is\partial_x U||_{L^\infty_x}\\ &\lesssim ||(x+2is\partial_x)\partial_x U||_{L^2_x} + s||\partial_x U||_{L^\infty_x} + ||(x+2is\partial_x)U||_{L^2_x}\end{aligned}$$

and the following bounds, the 2nd of which we establish later:

$$||\partial_x U(s)||_{L^\infty_x} \lesssim \langle s\rangle^{-\frac{1}{2}+\epsilon(\delta_2)},\ ||(x+2is\partial_x)\nabla U||_{L^2_x} \lesssim \langle s\rangle^{\frac{1}{2}+\epsilon(\delta_2)}$$

Now consider

$$P_{\geq a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U|^2]$$

Observe that

$$P_{\geq a}[\chi_{>0}|U|^2(s)] = P_{\geq a}\partial_x\triangle^{-1}[\partial_x(\chi_{>0})|U|^2(s)] + P_{\geq a}\partial_x\triangle^{-1}[\chi_{>0}\partial_x[|U|^2(s)]]$$

Note that

$$\partial_x(\chi_{>0})|U|^2(s) = \delta_0|U(0)|^2,$$

whence

$$||P_{\geq a}\partial_x\triangle^{-1}[\partial_x(\chi_{>0})|U|^2(s)]||_{L^1_x} \lesssim a^{-1}\langle s\rangle^{-3+\epsilon(\delta_3)}$$

Next, use that

$$is\partial_x[|U|^2] = is\partial_x U\bar{U} - U\overline{is\partial_x U} = (is\partial_x + \frac{x}{2})U\bar{U} - U\overline{(is\partial_x + \frac{x}{2})U}, \tag{5.3}$$

whence we get

$$||P_{\geq a}\partial_x\triangle^{-1}[\chi_{>0}\partial_x[|U|^2(s)]]||_{L^2_x} \lesssim a^{-1}\langle s\rangle^{-1}\langle s\rangle^{-\frac{1}{2}}$$

Arguing similarly for $P_{\geq a}[|U|^2]$, one gets

$$||P_{\geq a}[\chi_{>0}|U|^2]P_{\geq a}[|U|^2]||_{L^1_x} \lesssim a^{-2}\langle s\rangle^{-3}$$

Combining with the bound on $\begin{pmatrix}\bar{U}\\ U\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}$ given above, we can bound the whole expression by

$$\begin{aligned}&|\langle P_{\geq a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U|^2], \Pi_{(t^{-1000},t^{1000})}\langle \underline{e}, \chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle|\\ &\lesssim a^{-2}\langle s\rangle^{-3}\langle s\rangle^{\frac{1}{2}+\epsilon(\delta_2)}\langle t-s\rangle^{-\frac{3}{2}},\end{aligned}$$

which yields something almost integrable in s upon omitting the factor $\langle t-s\rangle^{-\frac{3}{2}}$ provided we choose $a = \langle s\rangle^{-\frac{3}{4}}$. This is good enough since by assumption $\delta_3 >> \delta_2$. We now consider the other extreme, the case of **low-low frequency interactions**, i. e. the expression

$$\int_0^\infty\int_0^{\frac{t}{2}}\langle\begin{pmatrix}P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2]\\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U^2|]\end{pmatrix}, s(\xi)e^{ix\xi}\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}]}(\xi)dsd\xi$$

We can express $\tilde{\mathcal{F}}...$ as before, and only the term $s(\xi)e^{ix\xi}\underline{e}$ in the Fourier basis matters. On account of the fact that

$$\langle\begin{pmatrix}P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2]\\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U^2|]\end{pmatrix}, s(\xi)e^{ix\xi}\underline{e}\rangle = \chi_{<a+O(1)}(\xi)\langle\begin{pmatrix}P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2]\\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U^2|]\end{pmatrix}, s(\xi)e^{ix\xi}\underline{e}\rangle,$$

we can estimate

$$\begin{aligned}&|\int_0^\infty\langle\begin{pmatrix}P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2]\\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U^2|]\end{pmatrix}, s(\xi)e^{ix\xi}\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}]}(\xi)d\xi|\\ &\lesssim a^2\langle s\rangle^{-\frac{3}{2}}\langle s\rangle\langle t-s\rangle^{-\frac{3}{2}},\end{aligned}$$

which for $a\sim\langle s\rangle^{-\frac{3}{4}}$ can be integrated in s to yield the bound $\langle t\rangle^{-\frac{3}{2}}$. Finally, we consider the **mixed case**, i. e. the expression

$$\int_0^\infty\int_0^{\frac{t}{2}}\langle\begin{pmatrix}P_{\geq a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2]\\ -P_{\geq a}[\chi_{>0}|U|^2(s)]P_{<a}[|U^2|]\end{pmatrix}, s(\xi)e^{ix\xi}\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}]}(\xi)dsd\xi$$

We proceed as before, simplifying $\tilde{\mathcal{F}}(...)$ by discarding the Schwartz term in the Fourier basis (as we may), and using the ordinary Plancherel's Theorem to translate this to the physical side. Arguing as before, we may do this by including a multiplier $\Pi_{(t^{-1000},t^{1000})}$ which is given by a kernel of L^1-mass $\lesssim \log t$. We indicate this by replacing the functions $P_{\geq a}[\chi_{>0}|U|^2(s)]$, $P_{<a}[|U|^2]$ by translates, $T_z P_{\geq a}[\chi_{>0}|U|^2(s)]$ and $T_z P_{<a}[|U|^2]$, where $(T_z f)(x) := f(x+z)$. The integration over z in the end will cost $\lesssim \log t$. Thus we now need to consider the following expression:

$$\langle T_z P_{\geq a}[\chi_{>0}|U|^2(s)] T_z P_{<a}[|U|^2], \langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle$$

We re-arrange the terms here:

$$\langle T_z P_{<a}[|U|^2], T_z P_{\geq a}[\chi_{>0}|U|^2(s)] \langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle\rangle$$

Revert to vectorial notation:

$$\langle \begin{pmatrix} T_z P_{<a}[|U|^2] \\ 0 \end{pmatrix}, \begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)] \langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle \\ 0 \end{pmatrix} \rangle$$

We break this into two portions:

$$\langle \chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2] \\ 0 \end{pmatrix}, \begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)] \langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle \\ 0 \end{pmatrix} \rangle$$

$$\langle \chi_{<0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2] \\ 0 \end{pmatrix}, \begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)] \langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle \\ 0 \end{pmatrix} \rangle$$

As these can be treated similarly, we consider only the first. Our first step consists in reducing the factor $\chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2] \\ 0 \end{pmatrix}$ to its dispersive part. Note that if we substitute $\langle \chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2] \\ 0 \end{pmatrix}, \xi_{k(j)} \rangle \eta_j$ for this expression instead, we can estimate

$$\begin{aligned} &\langle \langle \chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2] \\ 0 \end{pmatrix}, \xi_{k(j)} \rangle \eta_j, \begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)] \langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle \\ 0 \end{pmatrix} \rangle \\ &\lesssim \langle s \rangle^{-\frac{3}{2}} \langle t-s \rangle^{-\frac{3}{2}} \end{aligned}$$

This can be integrated in s to yield the upper bound $\lesssim \langle t \rangle^{-\frac{3}{2}}$. Now, with the left factor reduced to its dispersive part, invoking the distorted Plancherel's Theorem 2.3, we need to estimate

$$\sum_{\pm} \int_{-\infty}^{\infty} \mathcal{F}_{\pm}[\chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2] \\ 0 \end{pmatrix}](\xi) \overline{\tilde{\mathcal{F}}_{\pm}[\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)] \langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle \\ 0 \end{pmatrix}](\xi)} d\xi$$

We may and shall treat the case $+$, and omit the subscript for simplicity. As before, we need to subdivide the ξ-integration into two contributions, one from $(-\infty, 0]$, the other from $[0, \infty)$. We treat here the contribution from the latter, that from the former being more complicated and treated below. We decompose

$$\mathcal{F}[\chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2] \\ 0 \end{pmatrix}](\xi) = \mathcal{F}[\chi_{>0}(x) \begin{pmatrix} T_z[|U|^2] \\ 0 \end{pmatrix}](\xi) - \mathcal{F}[\chi_{>0}(x) \begin{pmatrix} T_z P_{\geq a}[|U|^2] \\ 0 \end{pmatrix}](\xi)$$

Substituting the 2nd summand results in an expression which can be treated like in the high-high case. Thus substitute the first summand on the right, $\mathcal{F}[\chi_{>0}(x) \begin{pmatrix} T_z[|U|^2] \\ 0 \end{pmatrix}](\xi)$. One explicitly writes out the Fourier transform, and may discard the contribution from the local part $\phi(x,\xi)$ of the Fourier basis, reasoning as before. Then one obtains the following expression:

$$\int_0^\infty \langle\chi_{>0}(x)\begin{pmatrix} T_z[|U|^2] \\ 0\end{pmatrix}, s(\xi)e^{ix\xi}\underline{e}\rangle \overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle \\ 0\end{pmatrix}](\xi)}d\xi$$

In this break $\chi_{>0}(x)\begin{pmatrix} T_z[|U|^2] \\ 0\end{pmatrix}$ into two parts, a large frequency and a small frequency part:

$$\chi_{>0}(x)\begin{pmatrix} T_z[|U|^2] \\ 0\end{pmatrix} = P_{\geq a}[\chi_{>0}(x)\begin{pmatrix} T_z[|U|^2] \\ 0\end{pmatrix}] + P_{<a}[\chi_{>0}(x)\begin{pmatrix} T_z[|U|^2] \\ 0\end{pmatrix}]$$

Consider the first summand on the right: one may differentiate the expression, replacing it by

$$\partial_x\triangle^{-1}P_{\geq a}[\delta_0(x)\begin{pmatrix} T_z[|U|^2] \\ 0\end{pmatrix}] + \partial_x\triangle^{-1}P_{\geq a}[\chi_{>0}(x)\begin{pmatrix} T_z\partial_x[|U|^2] \\ 0\end{pmatrix}]$$

Observe that the a^{-1} from the operator $\partial_x\triangle^{-1}P_{\geq a}$ is counteracted by the factor $s(\xi)$ above. In order to treat the contribution from the first summand, subdivide the interval $[a,\infty)$ into dyadic intervals, and sum. Thus we need to estimate

$$\sum_{2^j\geq a}\int_0^\infty \langle\begin{pmatrix} \partial_x\triangle^{-1}P_{2^j}[\delta_0(x)T_z[|U|^2] \\ 0\end{pmatrix}, s(\xi)e^{ix\xi}\underline{e}\rangle \overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle \\ 0\end{pmatrix}](\xi)}d\xi$$

We have

$$|\langle\begin{pmatrix} \partial_x\triangle^{-1}P_{2^j}[\delta_0(x)T_z[|U|^2] \\ 0\end{pmatrix}, s(\xi)e^{ix\xi}\underline{e}\rangle| \lesssim \min\{2^{-j},1\}s^{-1}$$

Moreover, we have

$$||\tilde{\mathcal{F}}[\begin{pmatrix} T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle \\ 0\end{pmatrix}](\xi)||_{L^2_\xi} \lesssim a^{-1}\langle s\rangle^{-\frac{3}{2}}\langle s\rangle^{\frac{1}{2}+\epsilon(\delta_2)}\langle t-s\rangle^{-\frac{3}{2}}$$

Thus, using Hölder's inequality, we have

$$|\int_0^\infty \langle\begin{pmatrix} \partial_x\triangle^{-1}P_{2^j}[\delta_0(x)T_z[|U|^2] \\ 0\end{pmatrix}, s(\xi)e^{ix\xi}\underline{e}\rangle \overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle \\ 0\end{pmatrix}](\xi)}d\xi|$$

$$\lesssim \min\{2^{-\frac{j}{2}},1\}\langle s\rangle^{-1}a^{-1}\langle s\rangle^{-\frac{3}{2}}\langle s\rangle^{\frac{1}{2}+\epsilon(\delta_2)}\langle t-s\rangle^{-\frac{3}{2}}$$

Summing over j costs at most $\log s$, whence substituting $a=\langle s\rangle^{-\frac{3}{4}}$ and integrating in s yields the upper bound $\lesssim \langle t\rangle^{-\frac{3}{2}+\delta_3}$, as desired. If, on the other hand, we substitute $\partial_x\triangle^{-1}P_{\geq a}[\chi_{>0}(x)T_z\partial_x[|U|^2]$, we argue just as for the high-high case. Now consider the expression

$$\int_0^\infty \langle P_{<a}[\chi_{>0}(x)\begin{pmatrix} T_z[|U|^2] \\ 0\end{pmatrix}], s(\xi)e^{ix\xi}\underline{e}\rangle \overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle \\ 0\end{pmatrix}](\xi)}d\xi$$

Estimate

$$||\langle P_{<a}[\chi_{>0}(x)\left(\begin{array}{c} T_z[|U|^2] \\ 0 \end{array}\right)], s(\xi)e^{ix\xi}\underline{e}\rangle||_{L^2_\xi} \lesssim a\langle s\rangle^{-\frac{1}{2}}$$

$$||\mathcal{F}[\left(\begin{array}{c} T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}\rangle \\ 0 \end{array}\right)](\xi)||_{L^2_\xi} \lesssim a^{-1}\langle s\rangle^{-\frac{3}{2}}\langle s\rangle^{\frac{1}{2}+\epsilon(\delta_2)}\langle t-s\rangle^{-\frac{3}{2}}$$

Putting these together results in the upper bound

$$\begin{aligned}
&|\int_0^\infty \langle P_{<a}[\chi_{>0}(x)\left(\begin{array}{c} T_z[|U|^2] \\ 0 \end{array}\right)], s(\xi)e^{ix\xi}\underline{e}\rangle \\
&\qquad\overline{\tilde{\mathcal{F}}[\left(\begin{array}{c} T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}\rangle \\ 0 \end{array}\right)](\xi)}d\xi| \\
&\lesssim a\langle s\rangle^{-\frac{1}{2}}a^{-1}\langle s\rangle^{-\frac{3}{2}}\langle s\rangle^{\frac{1}{2}+\epsilon(\delta_2)}\langle t-s\rangle^{-\frac{3}{2}} \lesssim \langle s\rangle^{-\frac{3}{2}+\epsilon(\delta_2)}\langle t-s\rangle^{-\frac{3}{2}},
\end{aligned}$$

which upon integration in s again yields the desired upper bound $\langle t\rangle^{-\frac{3}{2}+\delta_3}$. The case when the ξ-variable is restricted to $(-\infty, 0]$ in the mixed case will be treated further below.

Now we consider

$$\int_{-\infty}^0\int_0^{\frac{t}{2}}\mathcal{F}\left(\begin{array}{c}\chi_{>0}|U|^4(s)\\ -\chi_{>0}|U|^4(s)\end{array}\right)(\xi)\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}](\xi)}dsd\xi$$

Reformulate this as

$$\begin{aligned}
&\int_{-\infty}^0\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}\chi_{>0}|U|^4(s)\\ -\chi_{>0}|U|^4(s)\end{array}\right), (e^{ix\xi}-e^{-ix\xi})\underline{e}+(1+r(-\xi)e^{-ix\xi}\underline{e}+\phi(x,\xi)\rangle \\
&\qquad\qquad\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}](\xi)}dsd\xi
\end{aligned}$$

We can treat the contribution of $\phi(x,\xi)$ just as we did before. Also, note that $|1+r(-\xi)| = O(|\xi|)$ around $\xi = 0$, see Theorem 2.2, whence we can treat the contribution of this part just like we did for the transmission part before. The remaining part we break into a number of contributions:

$$\int_{-\infty}^0\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}P_{\geq a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U|^2(s)]\\ -P_{\geq a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U|^2(s)]\end{array}\right), (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)]}(\xi)dsd\xi$$

$$\int_{-\infty}^0\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}P_{<a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U|^2(s)]\\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{\geq a}[|U|^2(s)]\end{array}\right), (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)]}(\xi)dsd\xi$$

$$\int_{-\infty}^0\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}P_{\geq a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)]\\ -P_{\geq a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)]\end{array}\right), (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)]}(\xi)dsd\xi$$

$$\int_{-\infty}^0\int_0^{\frac{t}{2}}\langle\left(\begin{array}{c}P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)]\\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)]\end{array}\right), (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)]}(\xi)dsd\xi$$

Start with the first term in this list: write

$$\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)] = \langle\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right), [e^{ix\xi}+r(-\xi)e^{-ix\xi}]\underline{e}+\phi(x,\xi)\rangle$$

The contribution of $\phi(x,\xi)$ here is again straightforward, and left out. Now one proceeds as for the transmission part ($\xi \geq 0$) treated before, using the ordinary Plancherel's Theorem and introducing a multiplier $\Pi_{(t^{-1000}, t^{1000})}$.

Next, we consider the low-low frequency interaction, i. e. the expression

$$\int_{-\infty}^0\int_0^{t}\langle\left(\begin{array}{c}P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)]\\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)]\end{array}\right), (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)]}(\xi)dsd\xi$$

This calls for a different strategy than for the transmission part, since the Fourier basis in this regime doesn't vanish uniformly at $\xi = 0$. First, we observe that

$$\langle\left(\begin{array}{c} P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)] \\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)] \end{array}\right), (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle$$
$$= \chi_{<a+O(1)}(\xi)\langle\left(\begin{array}{c} P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)] \\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)] \end{array}\right), (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle$$

Hence we have

$$||\langle\left(\begin{array}{c} P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)] \\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)] \end{array}\right), (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle||_{L^1_\xi} \lesssim a\langle s\rangle^{-1}$$

Notice that putting $a = \langle s\rangle^{-\frac{3}{4}}$ is not quite good enough yet to counterbalance the loss of s arising when one extracts the $(t-s)^{-\frac{3}{2}}$-gain. This extra gain of $s^{-\frac{1}{4}}$ has to come from the 2nd factor $\tilde{\mathcal{F}}(\ldots)$. Write for $\xi < 0$

$$\tilde{\mathcal{F}}[\chi_{>0}\left(\begin{array}{c} \bar{U} \\ U \end{array}\right) \times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c} \phi \\ \psi \end{array}\right)_{dis}](\xi)$$
$$= \langle\chi_{>0}\left(\begin{array}{c} \bar{U} \\ U \end{array}\right) \times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c} \phi \\ \psi \end{array}\right)_{dis}, [e^{ix\xi} - e^{-ix\xi}]\underline{e} + (1 + r(-\xi))e^{-ix\xi}\underline{e} + \phi(x,\xi)\rangle$$

We first get rid of $(1 + r(-\xi))e^{-ix\xi}\underline{e} + \phi(x,\xi)$. Note that for $\xi \lesssim a$, we have

$$||\langle\chi_{>0}\left(\begin{array}{c} \bar{U} \\ U \end{array}\right) \times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c} \phi \\ \psi \end{array}\right)_{dis}, (1 + r(-\xi))e^{-ix\xi}\underline{e}\rangle||_{L^2_\xi} \lesssim a\langle s\rangle^{1+\epsilon(\delta_2)}\langle t-s\rangle^{-\frac{3}{2}}$$

Combining this with

$$||\langle\left(\begin{array}{c} P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)] \\ -P_{<a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)] \end{array}\right), (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle||_{L^2_\xi} \lesssim \langle s\rangle^{-\frac{3}{2}}$$

easily leads to the upper bound $\lesssim \langle s\rangle^{-\frac{5}{4+}}\langle t-s\rangle^{-\frac{3}{2}}$ for this contribution. Next, we have

$$||\langle\chi_{>0}\left(\begin{array}{c} \bar{U} \\ U \end{array}\right) \times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c} \phi \\ \psi \end{array}\right)_{dis}, \phi(x,\xi)\rangle||_{L^\infty_\xi} \lesssim \langle s\rangle^{-\frac{3}{2}+\delta_3}\langle t-s\rangle^{-\frac{3}{2}},$$

which similarly leads to an acceptable upper bound. We now reduce to estimating the expression

$$\int_{-\infty}^{\infty}[e^{ix\xi} - e^{-ix\xi}]\langle\underline{e}, \chi_{>0}\left(\begin{array}{c} \bar{U} \\ U \end{array}\right) \times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c} \phi \\ \psi \end{array}\right)_{dis}\rangle dx$$
$$= i\xi\int_0^{\infty}[e^{ix\xi} + e^{-ix\xi}]\langle\underline{e}, \int_x^{\infty}\chi_{>0}\left(\begin{array}{c} \bar{U}(s) \\ U(s) \end{array}\right) \times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c} \phi \\ \psi \end{array}\right)_{dis}(y)dy\rangle dx$$

In order to analyze the inner integral here, it appears useful to express U etc as Fresnel integrals, which makes the spatial oscillations visible. First, using the distorted Fourier transform, we write

$$\chi_{>0}e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c} \phi \\ \psi \end{array}\right)_{dis} = \sum_{\pm}\int_{-\infty}^{\infty} e^{\pm i(t-s)(\xi^2+1)}\chi_{>0}\sigma_3 e_{\pm}(x,\xi)\tilde{\mathcal{F}}_{\pm}(\left(\begin{array}{c} \phi \\ \psi \end{array}\right)(\xi)d\xi$$

Fix the $+$-sign here, the $-$-sign being treated accordingly; it is important here that the oscillatory part of $e_-(x,\xi)$ only has a lower component, i. e. $e_-(x,\xi) = e^{ix\xi}\sigma_1\underline{e} + \ldots$, where $\sigma_1 = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)$, $\underline{e} = \left(\begin{array}{c} 1 \\ 0 \end{array}\right)$, while the oscillatory part of $e_+(x,\xi)$ only has an upper component. Then we break the integral into two contributions:

$$\int_0^{\infty} e^{i(t-s)(\xi^2+1)}\chi_{>0}\sigma_3 e(x,\xi)\tilde{\mathcal{F}}(\left(\begin{array}{c} \phi \\ \psi \end{array}\right)(\xi)d\xi \tag{5.4}$$

$$\int_{-\infty}^{0} e^{i(t-s)(\xi^2+1)}\chi_{>0}\sigma_3 e(x,\xi)\tilde{\mathcal{F}}(\left(\begin{array}{c} \phi \\ \psi \end{array}\right)(\xi)d\xi \tag{5.5}$$

Write the first integral (5.4) as

$$\int_0^\infty e^{i(t-s)(\xi^2+1)}\chi_{>0}\sigma_3[s(\xi)e^{ix\xi}\underline{e}+\phi(x,\xi)]\tilde{\mathcal{F}}(\begin{pmatrix}\phi\\ \psi\end{pmatrix})(\xi)d\xi$$

Carry out an integration by parts in ξ, thereby replacing this by

$$\frac{1}{t-s}\int_0^\infty e^{i(t-s)(\xi^2+1)}\chi_{>0}(x)\sigma_3\partial_\xi([s(\xi)e^{ix\xi}\underline{e}+\phi(x,\xi)]\frac{\tilde{\mathcal{F}}(\begin{pmatrix}\phi\\ \psi\end{pmatrix})(\xi)}{\xi})d\xi$$

The contribution of $\phi(x,\xi)$ here is again negligible, as is easily seen. The worst case occurs when the derivative ∂_ξ falls on the phase $e^{ix\xi}$, costing a factor ix. Explicitly, this is the following expression:

$$\frac{ix}{t-s}\int_0^\infty e^{i(t-s)(\xi^2+1)}\chi_{>0}(x)\sigma_3 s(\xi)e^{ix\xi}\underline{e}\frac{\tilde{\mathcal{F}}(\begin{pmatrix}\phi\\ \psi\end{pmatrix})(\xi)}{\xi}d\xi$$

Break the ξ-integral into two, one over the interval $[0,t^{1000}]$, the other over its complement on $[0,\infty)$. On the latter, an additional integration by parts in ξ easily furnishes more than the needed gain in t. On the former interval, observe that we may interpret the integral

$$\int_0^\infty e^{i(t-s)\xi^2}\sigma_3[s(\xi)e^{ix\xi}\underline{e}\frac{\tilde{\mathcal{F}}(\begin{pmatrix}\phi\\ \psi\end{pmatrix})(\xi)}{\xi}\chi_{<t^{1000}}(\xi)d\xi$$

as a solution for the free Schroedinger equation, evaluated at time $t-s$, with initial data

$$g(x)=\int_0^\infty \sigma_3 s(\xi)e^{ix\xi}\underline{e}\frac{\tilde{\mathcal{F}}(\begin{pmatrix}\phi\\ \psi\end{pmatrix})(\xi)}{\xi}\chi_{<t^{1000}}(\xi)d\xi$$

The definition of $\tilde{\mathcal{F}}$ as well as further integrations by parts in ξ reveal that this decays like x^{-2} for large values of x, resulting in

$$||\int_0^\infty \sigma_3 s(\xi)e^{ix\xi}\underline{e}\frac{\tilde{\mathcal{F}}(\begin{pmatrix}\phi\\ \psi\end{pmatrix})(\xi)}{\xi}\chi_{<t^{1000}}(\xi)d\xi||_{L^1_x}\lesssim \log t$$

Thus we can now write

$$\frac{ix}{t-s}\int_0^\infty e^{i(t-s)(\xi^2+1)}\chi_{<t^{1000}}(\xi)\sigma_3 s(\xi)e^{ix\xi}\underline{e}\frac{\tilde{\mathcal{F}}(\begin{pmatrix}\phi\\ \psi\end{pmatrix})(\xi)}{\xi}d\xi=\frac{ie^{i(t-s)}x}{t-s}\frac{1}{\sqrt{t-s}}\int_{-\infty}^\infty e^{-\frac{(x-y)^2}{i(t-s)}}g(y)dy \tag{5.6}$$

Next, returning to (5.5), we consider the integral

$$\int_{-\infty}^0 e^{i(t-s)(\xi^2+1)}\chi_{>0}\sigma_3 e(x,\xi)\tilde{\mathcal{F}}(\begin{pmatrix}\phi\\ \psi\end{pmatrix})(\xi)d\xi$$

In the regime under consideration we can write $e(x,\xi)=[e^{ix\xi}-e^{-ix\xi}+(1+r(-\xi))e^{-ix\xi}]\underline{e}+\phi(x,\xi)$. We proceed as before, arriving (up to error terms handled as before) at the expression

$$\begin{aligned}&\frac{ix}{t-s}\int_{-\infty}^0 e^{i(t-s)(\xi^2+1)}\chi_{>0}(x)[e^{ix\xi}+e^{-ix\xi}]\underline{e}\frac{\tilde{\mathcal{F}}\begin{pmatrix}\phi\\ \psi\end{pmatrix}(\xi)}{\xi}d\xi\\ &=\frac{ix}{t-s}\frac{1}{\sqrt{t-s}}\int_{-\infty}^\infty e^{-\frac{(x-y)^2}{i(t-s)}}\tilde{g}(y)dy,\end{aligned} \tag{5.7}$$

where

$$\tilde{g}(y)=\int_{-\infty}^0 \chi_{>0}(x)[e^{ix\xi}+e^{-ix\xi}]\underline{e}\frac{\tilde{\mathcal{F}}\begin{pmatrix}\phi\\ \psi\end{pmatrix}(\xi)}{\xi}d\xi$$

Now substitute either (5.6) or (5.7) for the right hand factor in $\left(\begin{array}{c}\bar{U}\\ U\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}$. We replace the resulting $x\bar{U}$ by $s\partial_x\bar{U}$, upon using control over $||CU||_{L^2_x}$. Thus, if we substitute for example[58] (5.7), we need to estimate

$$\chi_{>0}(x)s\partial_x\bar{U}\frac{e^{i(t-s)}}{t-s}\int_{-\infty}^0 e^{i(t-s)\xi^2}[e^{ix\xi}+e^{-ix\xi}]\underline{e}\frac{\tilde{\mathcal{F}}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)(\xi)}{\xi}d\xi$$

We now schematically record the equation satisfied by $\partial_x\bar{U}$ as follows:

$$(-i\partial_t+\triangle)\partial_x\bar{U}=VU+V\bar{U}+V\partial_xU+V\partial_x\bar{U}+\ldots+\partial_x[|U|^4\bar{U}]$$

Here V denotes certain Schwartz functions whose fine structure is irrelevant. Thus we can write

$$\partial_x\bar{U}(s,x)=\int_{-\infty}^{\infty}\int_0^s\frac{1}{\sqrt{s-\lambda}}e^{-\frac{(x-y)^2}{i(s-\lambda)}}[VU(\lambda,y)+\partial_x[|U|^4\bar{U}(\lambda,.)]]d\lambda dy+\ldots$$

We claim that we may replace the local terms $VU(\lambda,y)$ by $\chi_{<s-s^{2\epsilon}}(\lambda)\chi_{<s^\epsilon}(y)VU(\lambda,y)$ for small $\epsilon>0$ (independent of δ_i etc), and the non-local term $\partial_x[|U|^4\bar{U}(\lambda,.)]$ by $\chi_{<s^{\frac{1}{2}}}(\lambda)\chi_{<s^{\frac{1}{2}}}(y)\partial_x[|U|^4\bar{U}(\lambda,.)]$. To see this, note that

$$||\int_{-\infty}^{\infty}\int_0^s\frac{1}{\sqrt{s-\lambda}}e^{-\frac{(x-y)^2}{i(s-\lambda)}}\chi_{\geq s^\epsilon}(y)VU(\lambda,y)d\lambda dy||_{L^2_x}\lesssim e^{-s^\epsilon}$$

$$||\int_{-\infty}^{\infty}\int_0^s\frac{1}{\sqrt{s-\lambda}}e^{-\frac{(x-y)^2}{i(s-\lambda)}}\chi_{<s^\epsilon}(y)\chi_{\geq s-s^{2\epsilon}}(\lambda)VU(\lambda,y)d\lambda dy||_{L^2_x}\lesssim\langle s\rangle^{-\frac{3}{2}}\langle s\rangle^{2\epsilon+\delta_3}$$

If one substitutes the corresponding terms in the Duhamel formula for U, $\bar{U}$ directly for the fifth factors U, $\bar{U}$ in

$$\int_0^{\frac{t}{2}}\langle e^{i(t-s)\mathcal{H}}\left(\begin{array}{c}|U|^4U(s,.)\\ -|U|^4\bar{U}(s,.)\end{array}\right)_{dis},\phi\rangle ds,$$

one easily bounds this contribution by $\lesssim\langle t\rangle^{-\frac{3}{2}}$. Similarly, we have

$$||\int_{-\infty}^{\infty}\int_0^s\frac{1}{\sqrt{s-\lambda}}e^{-\frac{(x-y)^2}{i(s-\lambda)}}\chi_{>s^{\frac{1}{2}}}(\lambda)|U|^4(\lambda,y)U(\lambda,y)d\lambda dy||_{L^2_x}\lesssim\langle s\rangle^{-\frac{1}{2}},$$

which leads to a similar conclusion upon substituting this integral for the last factors U, $\bar{U}$. Finally, note that on account of the pseudo-conformal conservation law, we have for $x>\lambda$

$$||U(\lambda,.)||_{L^2_x}\sim||\frac{\lambda}{x}\nabla U(\lambda,.)||_{L^2_x},$$

whence we can estimate

$$||\int_{-\infty}^{\infty}\int_0^{s^{\frac{1}{2}}}\frac{1}{\sqrt{s-\lambda}}e^{-\frac{(x-y)^2}{i(s-\lambda)}}|U(\lambda,y)|^4\chi_{>s^{\frac{1}{2}}}(y)U(y,\lambda)dy||_{L^2_x}\lesssim s^{-\frac{1}{2}}\int_0^{s^{\frac{1}{2}}}\lambda^{-2}\lambda d\lambda\lesssim\langle s\rangle^{-\frac{1}{2}}\log\langle s\rangle,$$

and the argument proceeds from here as before. This discussion justifies us in substituting

$$\partial_x\bar{U}(s,x)=\int_{-\infty}^{\infty}\int_0^s\frac{1}{\sqrt{s-\lambda}}e^{-\frac{(x-y)^2}{i(s-\lambda)}}[\chi_{<s-s^{2\epsilon}}\chi_{<s^\epsilon}(y)VU(\lambda,y)+\chi_{<s^{\frac{1}{2}}}(y)\chi_{<s^{\frac{1}{2}}}(\lambda)\partial_y[|U|^4\bar{U}(\lambda,y)]]d\lambda dy+\ldots$$

Next, write as before

$$\int_{-\infty}^0 e^{i(t-s)\xi^2}\chi_{<t^{1000}}(\xi)[e^{ix\xi}+e^{-ix\xi}]\underline{e}\frac{\tilde{\mathcal{F}}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)(\xi)}{\xi}d\xi=\frac{1}{\sqrt{t-s}}\int_{-\infty}^{\infty}e^{-\frac{(x-y')^2}{i(t-s)}}\tilde{g}(y')dy'$$

where

$$\tilde{g}(y')=\int_{-\infty}^0\chi_{<t^{1000}}(\xi)[e^{iy'\xi}+e^{-iy'\xi}]\underline{e}\frac{\tilde{\mathcal{F}}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)(\xi)}{\xi}d\xi$$

[58]The contribution of (5.6) is handled similarly.

We observe that we may include the cutoff $\chi_{<(t-s)^{\frac{1}{2}}}(y')$ in front of $\tilde{g}(y')$; this is on account of the estimate $|\tilde{g}(y')| \lesssim y'^{-2}\log t$. Finally, plugging these expressions into

$$\chi_{>0}(x)s\partial_x\bar{U}\frac{e^{i(t-s)}}{t-s}\int_{-\infty}^{0}e^{i(t-s)\xi^2}[e^{ix\xi}+e^{-ix\xi}]\underline{e}\frac{\tilde{\mathcal{F}}\begin{pmatrix}\phi\\ \psi\end{pmatrix}(\xi)}{\xi}d\xi,$$

and keeping in mind that our point of departure was the expression

$$\int_x^\infty \chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}(y)dy,$$

we arrive at terms of the following form:

$$\frac{s}{(t-s)^{\frac{3}{2}}}\int_{x_0}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\chi_{>0}(x)\int_0^s\frac{1}{\sqrt{s-\lambda}}e^{-\frac{(x-y)^2}{i(s-\lambda)}}\chi_{<s^\epsilon}(y)\chi_{<s-s^{2\epsilon}}(\lambda)VU(\lambda,y)e^{-\frac{(x-y')^2}{i(t-s)}} \chi_{<(t-s)^{\frac{1}{2}}}(y')\tilde{g}(y')d\lambda dydy'dx, \tag{5.8}$$

$$\frac{s}{(t-s)^{\frac{3}{2}}}\int_{x_0}^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\chi_{>0}(x)\int_0^s\frac{1}{\sqrt{s-\lambda}}e^{-\frac{(x-y)^2}{i(s-\lambda)}}|U(\lambda,y)|^4\chi_{<s^{\frac{1}{2}}}(y)\chi_{<s^{\frac{1}{2}}}(\lambda)U(y,\lambda)e^{-\frac{(x-y')^2}{i(t-s)}} \chi_{<(t-s)^{\frac{1}{2}}}(y')\tilde{g}(y')d\lambda dydy'dx, \tag{5.9}$$

where x_0 ranges over $[0,\infty]$, plus similar terms which can be treated identically. Write

$$e^{-\frac{(x-y)^2}{i(s-\lambda)}}e^{-\frac{(x-y')^2}{i(t-s)}}=e^{+i[(\frac{1}{s-\lambda}+\frac{1}{t-s})x^2-\frac{2xy}{s-\lambda}-\frac{2xy'}{t-s}]}e^{+\frac{iy^2}{s-\lambda}+\frac{iy'^2}{t-s}}$$

This can be rewritten as

$$e^{i(x\sqrt{\frac{1}{s-\lambda}+\frac{1}{t-s}}-y_1)^2}e^{iy_2}$$

for certain functions $y_{1,2}(y,y',s,\lambda,t)$. Our restrictions in either term (5.8) or (5.9) ensure that $y_1=O(1)$. Carrying out the x-integration, we obtain

$$\frac{1}{\sqrt{s-\lambda}}\int_{x_0}^\infty e^{i(x\sqrt{\frac{1}{s-\lambda}+\frac{1}{t-s}}-y_1)^2}dx=\frac{1}{\sqrt{s-\lambda}}(\frac{1}{s-\lambda}+\frac{1}{t-s})^{-\frac{1}{2}}S(x_0\sqrt{\frac{1}{s-\lambda}+\frac{1}{t-s}}-y_1)$$

where $S(y)=\int_y^\infty e^{ix^2}dx=\frac{e^{iy^2}}{y}+O(y^{-2})$ as $y\to+\infty$. Finally, we need to estimate

$$\xi\int_0^\infty[e^{ix_0\xi}+e^{-ix_0\xi}]\frac{1}{\sqrt{s-\lambda}}(\frac{1}{s-\lambda}+\frac{1}{t-s})^{-\frac{1}{2}}S(x_0\sqrt{\frac{1}{s-\lambda}+\frac{1}{t-s}}-y_1)dx_0$$

Here it is important that y_1 be uniformly bounded. The oscillatory nature of $S(y)$ allows us to bound this integral by

$$\lesssim|\xi|\sqrt{\langle s\rangle}\lesssim\langle s\rangle^{-\frac{1}{4}},$$

as desired. The remaining integrations over y,y' are straightforward to carry out on account of the integrability of the functions $g(y)$, $g(y')$. One pays $(\log t)^2$, which is irrelevant. This finally completes treating the low-low case.

We proceed to the mixed frequency case. This is the expression

$$\int_{-\infty}^0\int_0^{\frac{t}{2}}\langle\begin{pmatrix}P_{\geq a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)]\\ -P_{\geq a}[\chi_{>0}|U|^2(s)]P_{<a}[|U|^2(s)]\end{pmatrix},(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}](\xi)}dsd\xi$$

We first employ the ordinary Plancherel's Theorem to replace this (up to negligible error terms) by an expression

$$\langle T_zP_{<a}[|U|^2(s)],T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle\rangle,$$

where we have to integrate over z in the end which will cost $\log t$, as before. Express this in vectorial form as

$$\langle \begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}, \begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \rangle \\ 0 \end{pmatrix} \rangle,$$

Decompose this into the following two terms:

$$\langle \chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}, \begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle \\ 0 \end{pmatrix} \rangle,$$

$$\langle \chi_{<0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}, \begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle \\ 0 \end{pmatrix} \rangle,$$

These being treated similarly, we treat the first term: commence by replacing $\chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}$ by its dispersive part. This is done as in the mixed frequency case treated earlier. Then use the distorted Plancherel's Theorem 2.3, which produces

$$\int_{-\infty}^{\infty} \mathcal{F}[\chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}](\xi)$$
$$\overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle \\ 0 \end{pmatrix}](\xi)}\, d\xi$$

Divide this into the integral over $(-\infty, 0]$ as well as the integral over $[0, \infty)$. We treat the more difficult former case, the latter already having been dealt with in the preceding. We recast this as

$$\int_{-\infty}^{0} \langle \chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}, [e^{ix\xi} - e^{-ix\xi} + (1 + r(-\xi))e^{-ix\xi}]\underline{e} + \phi(x, \xi) \rangle$$
$$\overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle \\ 0 \end{pmatrix}](\xi)}\, d\xi$$

The contributions of $1 + r(-\xi)$ and $\phi(x, \xi)$ are straightforward, and handled as in the preceding. We then need to estimate the following two contributions:

$$\int_{-\infty}^{0} \langle P_{\geq a}[\chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}], [e^{ix\xi} - e^{-ix\xi}]\underline{e} \rangle$$
$$\overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle \\ 0 \end{pmatrix}](\xi)}\, d\xi$$

$$\int_{-\infty}^{0} \langle P_{<a}[\chi_{>0}(x) \begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}], [e^{ix\xi} - e^{-ix\xi}]\underline{e} \rangle$$
$$\overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0} \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis} \rangle \\ 0 \end{pmatrix}](\xi)}\, d\xi$$

Consider the first of these terms: it is straightforward to replace $P_{\geq a}[\chi_{>0}(x)\begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}]$ by $P_{\geq a}[\chi_{>0}(x)\begin{pmatrix} T_z[|U|^2(s)] \\ 0 \end{pmatrix}]$, by arguing as for the high-high case. Then we replace this term by

$$\int_{-\infty}^{0}\langle\partial_x\triangle^{-1}P_{\geq a}\partial_x[\chi_{>0}(x)\begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}],[e^{ix\xi}-e^{-ix\xi}]\underline{e}\rangle$$
$$\overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle \\ 0 \end{pmatrix}](\xi)}d\xi$$

First let the inner derivative ∂_x fall onto the factor $\chi_{>0}(x)$. This results in

$$\int_{-\infty}^{0}\langle\partial_x\triangle^{-1}P_{\geq a}[\delta_0(x)\begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}],[e^{ix\xi}-e^{-ix\xi}]\underline{e}\rangle$$
$$\overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle \\ 0 \end{pmatrix}](\xi)}d\xi$$

In order to estimate this, we decompose it further into two contributions:

$$\int_{-\infty}^{0}\langle\partial_x\triangle^{-1}P_{\langle s\rangle^{-\frac{1}{2}}\geq\cdot\geq a}[\delta_0(x)\begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}],[e^{ix\xi}-e^{-ix\xi}]\underline{e}\rangle$$
$$\overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle \\ 0 \end{pmatrix}](\xi)}d\xi$$

$$\int_{-\infty}^{0}\langle\partial_x\triangle^{-1}P_{>\langle s\rangle^{-\frac{1}{2}}}[\delta_0(x)\begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}],[e^{ix\xi}-e^{-ix\xi}]\underline{e}\rangle$$
$$\overline{\tilde{\mathcal{F}}[\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle \\ 0 \end{pmatrix}](\xi)}d\xi$$

Freeze the frequency of $P_{\langle s\rangle^{-\frac{1}{2}}\geq\cdot\geq a}[\delta_0(x)\begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}]$ to dyadic size $\sim b$. By Bernstein's inequality we get

$$||\triangle^{-1}\partial_x P_b[\delta_0(x)\begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}]||_{L_x^2}\lesssim b^{-\frac{1}{2}}\langle s\rangle^{-1+\epsilon(\delta_2)} \tag{5.10}$$

Next, note that

$$\tilde{\mathcal{F}}[\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}\rangle \\ 0 \end{pmatrix}](\xi)$$
$$=\int_0^\infty\langle\overline{[(e^{ix\xi}-e^{-ix\xi}+(1+r(-\xi)e^{-ix\xi}))\underline{e}+\phi(x,\xi)]},$$
$$\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\begin{pmatrix}\bar{U}(s)\\ U(s)\end{pmatrix}\times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_{dis}\rangle \\ 0 \end{pmatrix}\rangle dx$$

We treat here the most difficult contribution which comes as usual from $e^{ix\xi}-e^{-ix\xi}$. Carrying out an integration by parts, we have to estimate the following terms:

$$\int_0^\infty \xi[e^{ix\xi}+e^{-ix\xi}]\left(\begin{array}{c} T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\int_x^\infty\langle\underline{e},\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}(s,y)\rangle \\ 0\end{array}\right)dydx$$

$$\int_0^\infty [e^{ix\xi}-e^{-ix\xi}]\left(\begin{array}{c} T_zP_{\geq a}\partial_x[\chi_{>0}|U|^2(s)]\int_x^\infty\langle\underline{e},\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}(s,y)\rangle \\ 0\end{array}\right)dydx$$

Our calculations for the low-low case above have taught us that we may assume[59]

$$|\int_x^\infty\langle\underline{e},\chi_{>0}\left(\begin{array}{c}\bar{U}(s,y)\\ U(s,y)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}\rangle dy|\lesssim\langle s\rangle\langle t-s\rangle^{-\frac{3}{2}}$$

Using that $|\xi|\sim b$, the ordinary Plancherel's Theorem then implies that we have

$$||\int_0^\infty \xi[e^{ix\xi}+e^{-ix\xi}]\left(\begin{array}{c} T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\int_x^\infty\langle\underline{e},\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}(s,y)\rangle \\ 0\end{array}\right)dydx||_{L^2_\xi}$$
$$\lesssim\langle s\rangle\langle t-s\rangle^{-\frac{3}{2}}b||T_zP_{\geq a}[\chi_{>0}|U|^2(s)]||_{L^2_x}\lesssim b\langle s\rangle^{-\frac{3}{4}}\langle s\rangle\langle t-s\rangle^{-\frac{3}{2}}$$

The contribution of the term

$$\int_{-\infty}^0 [e^{ix\xi}-e^{-ix\xi}]\left(\begin{array}{c} T_zP_{\geq a}\partial_x[\chi_{>0}|U|^2(s)]\int_x^\infty\langle\underline{e},\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}(s,y)\rangle \\ 0\end{array}\right)dydx$$

is handled similarly, arguing as in the high-high case, using (5.3). Combining this with the bound (5.10) from before, we estimate

$$|\int_{-\infty}^0\langle\partial_x\triangle^{-1}P_b[\delta_0(x)\left(\begin{array}{c}T_zP_{<a}[|U|^2(s)]\\ 0\end{array}\right)],[e^{ix\xi}-e^{-ix\xi}]\underline{e}\rangle$$
$$\overline{\tilde{\mathcal{F}}[\left(\begin{array}{c}T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}\rangle\\ 0\end{array}\right)](\xi)}d\xi|$$
$$\lesssim b^{-\frac{1}{2}}\langle s\rangle^{-1+\epsilon(\delta_2)}b\langle s\rangle^{\frac{1}{4}}\langle t-s\rangle^{-\frac{3}{2}}$$

Summing over all dyadic b with $a<b<\langle s\rangle^{-\frac{1}{2}}$ and integrating over s results in the bound $\lesssim\langle t\rangle^{-\frac{3}{2}+\delta_3}$. Next, we consider the contribution of

$$\int_{-\infty}^0\langle\partial_x\triangle^{-1}P_{>\langle s\rangle^{-\frac{1}{2}}}[\delta_0(x)\left(\begin{array}{c}T_zP_{<a}[|U|^2(s)]\\ 0\end{array}\right)],[e^{ix\xi}-e^{-ix\xi}]\underline{e}\rangle$$
$$\overline{\tilde{\mathcal{F}}[\left(\begin{array}{c}T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\langle\underline{e},\chi_{>0}\left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}\rangle\\ 0\end{array}\right)](\xi)}d\xi$$

---

[59]More precisely, we may write $\left(\begin{array}{c}U\\ \bar{U}\end{array}\right)$ as the sum of two functions, one of which leads to a trivially estimable contribution, while the other satisfies the above inequality.

We may again essentially replace $\tilde{\mathcal{F}}$ by the ordinary Fourier transform, and invoke the ordinary Plancherel's Theorem to replace this by (up to negligible errors)

$$\langle \partial_x \triangle^{-1} P_{>\langle s\rangle^{-\frac{1}{2}}}[\delta_0(x)\begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}],$$

$$\Pi_{(t^{-1000},t^{1000})}\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0}\begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}\rangle \\ 0 \end{pmatrix}\rangle,$$

where $\Pi_{(t^{-1000},t^{1000})}$ is as in the discussion of the high-high case. We bound this by

$$\lesssim ||\partial_x \triangle^{-1} P_{>\langle s\rangle^{-\frac{1}{2}}}[\delta_0(x)\begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}]||_{L^1_x}$$

$$||\Pi_{(t^{-1000},t^{1000})}\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0}\begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}\rangle \\ 0 \end{pmatrix}||_{L^\infty_x}$$

We can bound the preceding expression by

$$\lesssim \log t \langle s\rangle^{\epsilon(\delta_2)}\langle s\rangle^{\frac{1}{2}}\langle s\rangle^{-1}\langle s\rangle^{-1}\langle s\rangle^{\frac{1}{2}}\langle t-s\rangle^{-\frac{3}{2}},$$

which upon integration over s leads to an acceptable bound. Thus in order to complete the discussion for the case $s<\frac{t}{2}$, we need to estimate the expression

$$\int_{-\infty}^{0}\frac{\langle P_{<a}[\chi_{>0}(x)\begin{pmatrix} T_z P_{<a}[|U|^2(s)] \\ 0 \end{pmatrix}], [e^{ix\xi}-e^{-ix\xi}]\underline{e}\rangle}{\tilde{\mathcal{F}}[\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0}\begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}\rangle \\ 0 \end{pmatrix}](\xi)d\xi}$$

Keep in mind that we put $a=\langle s\rangle^{-\frac{3}{4}}$. As usual we simplify $\tilde{\mathcal{F}}[...]$ and carry out an integration by parts, replacing this by

$$\xi\langle [e^{ix\xi}+e^{-ix\xi}]\underline{e}, [\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0}\int_x^\infty \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}\rangle \\ 0 \end{pmatrix}\rangle$$

$$\langle [e^{ix\xi}-e^{-ix\xi}]\underline{e}, [\begin{pmatrix} T_z P_{\geq a}\partial_x[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0}\int_x^\infty \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}\rangle \\ 0 \end{pmatrix}\rangle$$

Consider the first of these terms. The 2nd is treated similarly, using (5.3). We have

$$||\chi_{<\langle s\rangle^{-\frac{3}{4}}}(\xi)\xi\langle [e^{ix\xi}+e^{-ix\xi}]\underline{e}, [\begin{pmatrix} T_z P_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0}\int_x^\infty \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}\rangle \\ 0 \end{pmatrix}\rangle||_{L^\infty_\xi}$$

$$\lesssim \langle s\rangle^{-\frac{3}{4}}||T_z P_{\geq a}[\chi_{>0}|U|^2(s)]||_{L^2_x}||\chi_{>0}(x)\int_x^\infty \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}||_{L^2_x}$$

From our treatment of the low-low case we may assume that

$$||\chi_{>0}(x)\int_x^\infty \begin{pmatrix} \bar{U}(s) \\ U(s) \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*}\begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}||_{L^2_x} \lesssim \langle s\rangle\langle s\rangle^{\frac{1}{4}}\langle t-s\rangle^{-\frac{3}{2}},$$

More precisely we may decompose $\begin{pmatrix} U \\ \bar{U} \end{pmatrix}$ into two constituents one of which upon substitution into the original quintilinear expression immediately yields the desired estimate, while the other constituent satisfies the above estimate, see the discussion of the low-low case. We also have

$$||T_z P_{\geq a}[\chi_{>0}|U|^2(s)]||_{L^2_x} \lesssim a^{-1}\langle s\rangle^{-\frac{3}{2}} \lesssim \langle s\rangle^{-\frac{3}{4}}$$

Combining this with

$$||\langle P_{<a}[\chi_{>0}(x)\left(\begin{array}{c} T_zP_{<a}[|U|^2(s)] \\ 0 \end{array}\right)], [e^{ix\xi}-e^{-ix\xi}]\underline{e}\rangle||_{L^1_\xi} \lesssim \langle s\rangle^{-\frac{3}{4}},$$

we can bound

$$\begin{aligned}&|\int_{-\infty}^0 \langle P_{<a}[\chi_{>0}(x)\left(\begin{array}{c} T_zP_{<a}[|U|^2(s)] \\ 0 \end{array}\right)], [e^{ix\xi}-e^{-ix\xi}]\underline{e}\rangle \\ &\overline{\xi\langle e^{ix\xi}+e^{-ix\xi}, [\left(\begin{array}{c} T_zP_{\geq a}[\chi_{>0}|U|^2(s)]\langle \underline{e}, \chi_{>0}\int_x^\infty \left(\begin{array}{c}\bar{U}(s)\\ U(s)\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}\rangle \\ 0\end{array}\right)]\rangle} d\xi| \\ &\lesssim \langle s\rangle^{\frac{5}{4}}\langle s\rangle^{-\frac{3}{4}}\langle s\rangle^{-\frac{3}{4}}\langle s\rangle^{-\frac{3}{4}}\langle t-s\rangle^{-\frac{3}{2}},\end{aligned}$$

which is again as desired.

Case B: $s \geq \frac{t}{2}$. The procedure here is basically identical to the preceding case **A**, so we shall be relatively short here: one divides into the cases

$$\int_{\frac{t}{2}}^t \langle \left(\begin{array}{c} \chi_{>0}|U|^4U(s,.) \\ -\chi_{>0}|U|^4\bar{U}(s,.) \end{array}\right), e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}\rangle ds$$

$$\int_{\frac{t}{2}}^t \langle \left(\begin{array}{c} \chi_{<0}|U|^4U(s,.) \\ -\chi_{<0}|U|^4\bar{U}(s,.) \end{array}\right), e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}\rangle ds$$

Both being treated similarly, we shall only consider the first term. We easily reduce $\left(\begin{array}{c} \chi_{>0}|U|^4U(s,.) \\ -\chi_{>0}|U|^4\bar{U}(s,.) \end{array}\right)$ to its dispersive part: note that

$$|\langle\langle \left(\begin{array}{c} \chi_{>0}|U|^4U(s,.) \\ -\chi_{>0}|U|^4\bar{U}(s,.) \end{array}\right), \xi_{k(j)}\rangle\eta_j, e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}\rangle| \lesssim s^{-6+\epsilon}(t-s)^{-\frac{3}{2}},$$

which is significantly better than what we need. Now use the distorted Plancherel's Theorem to rewrite what remains as

$$\sum_{\pm}\int_{-\infty}^\infty \int_{\frac{t}{2}}^t \mathcal{F}_\pm\left(\begin{array}{c} \chi_{>0}|U|^4 \\ -\chi_{>0}|U|^4 \end{array}\right)(\xi)\overline{\tilde{\mathcal{F}}_\pm[\left(\begin{array}{c}\bar{U}\\ U\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}](\xi)}ds d\xi$$

We consider here the case $+$ and $\xi \in [0,\infty)$ and how one has to modify the argument in case **A** to get the desired estimate. Analogous modifications will then also give the result for $\xi \in (-\infty, 0]$. Write (leaving out the subscript)

$$\begin{aligned}&\int_0^\infty \int_{\frac{t}{2}}^t \mathcal{F}\left(\begin{array}{c} \chi_{>0}|U|^4 \\ -\chi_{>0}|U|^4 \end{array}\right)(\xi)\overline{\tilde{\mathcal{F}}[\left(\begin{array}{c}\bar{U}\\ U\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}](\xi)}ds d\xi \\ &= \int_0^\infty \int_{\frac{t}{2}}^t \langle \left(\begin{array}{c} \chi_{>0}|U|^4 \\ -\chi_{>0}|U|^4 \end{array}\right), s(\xi)e^{ix\xi}+\sigma_3\phi(x,\xi)\rangle\overline{\tilde{\mathcal{F}}[\left(\begin{array}{c}\bar{U}\\ U\end{array}\right)\times e^{-i(t-s)\mathcal{H}^*}\left(\begin{array}{c}\phi\\ \psi\end{array}\right)_{dis}](\xi)}ds d\xi\end{aligned}$$

The contribution of the local term $\phi(x,\xi)$ is again easy to handle. As usual invoke the decomposition

$$\int_0^\infty \int_{\frac{t}{2}}^t \langle \begin{pmatrix} \chi_{>0}|U|^4 \\ -\chi_{>0}|U|^4 \end{pmatrix}, s(\xi)e^{ix\xi}\rangle \overline{\tilde{\mathcal{F}}[\begin{pmatrix} \bar{U} \\ U \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}](\xi)} ds d\xi$$

$$= \int_0^\infty \int_{\frac{t}{2}}^t \langle \begin{pmatrix} P_{\geq a}[\chi_{>0}|U|^2]P_{\geq a}[|U|^2] \\ -P_{\geq a}[\chi_{>0}|U|^2]P_{\geq a}[|U|^2] \end{pmatrix}, s(\xi)e^{ix\xi}\rangle \overline{\tilde{\mathcal{F}}[\begin{pmatrix} \bar{U} \\ U \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}](\xi)} ds d\xi$$

$$+ \int_0^\infty \int_{\frac{t}{2}}^t \langle \begin{pmatrix} P_{\geq a}[\chi_{>0}|U|^2]P_{<a}[|U|^2] \\ -P_{\geq a}[\chi_{>0}|U|^2]P_{<a}[|U|^2] \end{pmatrix}, s(\xi)e^{ix\xi}\rangle \overline{\tilde{\mathcal{F}}[\begin{pmatrix} \bar{U} \\ U \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}](\xi)} ds d\xi$$

$$+ \int_0^\infty \int_{\frac{t}{2}}^t \langle \begin{pmatrix} P_{<a}[\chi_{>0}|U|^2]P_{\geq a}[|U|^2] \\ -P_{<a}[\chi_{>0}|U|^2]P_{\geq a}[|U|^2] \end{pmatrix}, s(\xi)e^{ix\xi}\rangle \overline{\tilde{\mathcal{F}}[\begin{pmatrix} \bar{U} \\ U \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}](\xi)} ds d\xi$$

$$+ \int_0^\infty \int_{\frac{t}{2}}^t \langle \begin{pmatrix} P_{<a}[\chi_{>0}|U|^2]P_{<a}[|U|^2] \\ -P_{<a}[\chi_{>0}|U|^2]P_{<a}[|U|^2] \end{pmatrix}, s(\xi)e^{ix\xi}\rangle \overline{\tilde{\mathcal{F}}[\begin{pmatrix} \bar{U} \\ U \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}](\xi)} ds d\xi$$

We consider here the first term. Use that for $\xi > 0$ we have

$$\tilde{\mathcal{F}}[\chi_{>0}(x) \begin{pmatrix} \bar{U} \\ U \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}](\xi) = \langle \chi_{>0}(x) \begin{pmatrix} \bar{U} \\ U \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}, s(\xi)e^{ix\xi} + \phi(x,\xi)\rangle$$

The contribution from $\phi(x,\xi)$ is easy to handle: note that

$$||\langle \sigma_3\phi(x,\xi), \chi_{>0}(x) \begin{pmatrix} \bar{U} \\ U \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}\rangle||_{L^2_\xi} \lesssim \langle t-s\rangle^{-\frac{3}{2}} s^{-\frac{3}{2}+\epsilon},$$

which is clearly good enough to close everything. Thus we now need to consider

$$\int_0^\infty \int_{\frac{t}{2}}^t \langle \begin{pmatrix} P_{\geq a}[\chi_{>0}|U|^2]P_{\geq a}[|U|^2] \\ -P_{\geq a}[\chi_{>0}|U|^2]P_{\geq a}[|U|^2] \end{pmatrix}, s(\xi)e^{ix\xi}\underline{e}\rangle \overline{\langle \chi_{>0}(x) \begin{pmatrix} \bar{U} \\ U \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}, s(\xi)e^{ix\xi}\underline{e}\rangle} ds d\xi$$

Using the ordinary Plancherel's Theorem, we replace this by (up to negligible error terms)

$$\int_0^\infty \int_{\frac{t}{2}}^t \langle P_{\geq a}[\chi_{>0}|U|^2]P_{\geq a}[|U|^2], \Pi_{(t^{-1000},t^{1000})}\langle \underline{e}, \chi_{>0}(x) \begin{pmatrix} \bar{U} \\ U \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}\rangle$$

This is all like in the case $s < \frac{t}{2}$. At this point, though, we don't pull down the full power of $(t-s)^{-\frac{3}{2}}$, but only $(t-s)^{-\frac{1}{2}}$, which costs nothing in terms of weights. In other words, we estimate

$$|\Pi_{(t^{-1000},t^{1000})}\langle \underline{e}, \chi_{>0}(x) \begin{pmatrix} \bar{U} \\ U \end{pmatrix} \times e^{-i(t-s)\mathcal{H}^*} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_{dis}\rangle| \lesssim (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}}$$

Then use again that

$$||P_{\geq a}[\chi_{>0}|U|^2]P_{\geq a}[|U|^2]||_{L^1_x} \lesssim a^{-2}s^{-3}$$

Putting these together and integrating up over $s > \frac{t}{2}$ yields the upper bound $t^{-\frac{3}{2}}$ up to an arbitrarily small exponential error independent of δ_3. The remaining terms above follow by similar modifications from the arguments for the case $s < \frac{t}{2}$, and are omitted. This establishes the strong local dispersive estimate up to demonstrating the bound $||C\partial_x U(s,.)||_{L^2_x} \lesssim s^{\frac{1}{2}+\epsilon(\delta_2)}$, which we shall do in the next subsection.

5.2. **Establishing the pseudo-conformal almost conservation law.** We now demonstrate that $\sup_{T>t\geq 0}||C\tilde{U}(t,.)||_{L^2_x} \leq \frac{\Lambda}{100}\delta$, provided we have already improved all the estimates of (3.48) *without the norm* $\sup_{T\geq s\geq 0}||CU(s,y)||_{L^2_y}$ to have righthand side $\frac{\Lambda}{1000}\delta$; also, assume that we have already improved the bound on (3.47)[60] to be $\leq \frac{\Lambda\delta^2}{1000}$, say. It is clear that we may alternatively prove this estimate for $\begin{pmatrix} U \\ \bar{U} \end{pmatrix}(t,.)$,

[60] These will later be improved independently of this subsection, of course.

in light of the already bootstrapped bounds. Write[61]

$$i\partial_t\langle C\begin{pmatrix} U\\ \bar{U}\end{pmatrix},\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\rangle=\langle i\dot{C}\begin{pmatrix} U\\ \bar{U}\end{pmatrix}+iC\partial_t\begin{pmatrix} U\\ \bar{U}\end{pmatrix},\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\rangle-\langle C\begin{pmatrix} U\\ \bar{U}\end{pmatrix},i\partial_t\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\rangle,$$

where

$$C=\begin{pmatrix} (x-2tp)^2 & 0\\ 0 & (x+2tp)^2\end{pmatrix},\ p=-i\partial_x$$

Then write the equation for $\begin{pmatrix} U\\ \bar{U}\end{pmatrix}$ schematically as follows: with $\mathcal{H}_0=\begin{pmatrix} \partial_y^2-1 & 0\\ 0 & 1-\partial_y^2\end{pmatrix}$

$$i\partial_t\begin{pmatrix} U\\ \bar{U}\end{pmatrix}+\mathcal{H}_0\begin{pmatrix} U\\ \bar{U}\end{pmatrix}=-\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2\nu^2\tilde{\phi}_0^4e^{2i(\Psi-\Psi_\infty)}\\ -2\nu^2\tilde{\phi}_0^4e^{-2i(\Psi-\Psi_\infty)} & -3\nu^2\tilde{\phi}_0^4\end{pmatrix}\begin{pmatrix} U\\ \bar{U}\end{pmatrix}+\dot{\pi}\partial_\pi W+N(U,\pi)$$

where $\tilde{\phi}=\phi(\nu(t)(.-\lambda_\infty[\mu-\mu_\infty](t))$. If we substitute this back into the preceding and expand $\dot{C}$, we obtain the relation

$$\begin{aligned}i\partial_t\langle C\begin{pmatrix} U\\ \bar{U}\end{pmatrix},\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\rangle=&\langle\begin{pmatrix} 8itp^2-2-4ixp & 0\\ 0 & 2+8itp^2+4ixp\end{pmatrix}\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\\ &+C(-\mathcal{H}_0\begin{pmatrix} U\\ \bar{U}\end{pmatrix}-\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2\nu^2\tilde{\phi}_0^4e^{2i(\Psi-\Psi_\infty)}\\ -2\nu^2\tilde{\phi}_0^4e^{-2i(\Psi-\Psi_\infty)} & -3\nu^2\tilde{\phi}_0^4\end{pmatrix}\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\\ &+\dot{\pi}\partial_\pi W+N(U,\pi)),\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\rangle\\ &+\langle C\begin{pmatrix} U\\ \bar{U}\end{pmatrix},\mathcal{H}_0\begin{pmatrix} U\\ \bar{U}\end{pmatrix}+\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2\nu^2\tilde{\phi}_0^4e^{2i(\Psi-\Psi_\infty)}\\ -2\nu^2\tilde{\phi}_0^4e^{-2i(\Psi-\Psi_\infty)} & -3\nu^2\tilde{\phi}_0^4\end{pmatrix}\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\\ &-\dot{\pi}\partial_\pi W-N(U,\pi)\rangle\end{aligned}$$

One can simplify the right hand side to the following:

$$\begin{aligned}&\langle -C\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2\nu^2\tilde{\phi}_0^4e^{2i(\Psi-\Psi_\infty)}\\ -2\nu^2\tilde{\phi}_0^4e^{-2i(\Psi-\Psi_\infty)} & -3\nu^2\tilde{\phi}_0^4\end{pmatrix}\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\\ &+\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & -2\nu^2\tilde{\phi}_0^4e^{2i(\Psi-\Psi_\infty)}\\ 2\nu^2\tilde{\phi}_0^4e^{-2i(\Psi-\Psi_\infty)} & -3\nu^2\tilde{\phi}_0^4\end{pmatrix}C\begin{pmatrix} U\\ \bar{U}\end{pmatrix},\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\rangle\\ &+\langle C(\dot{\pi}\partial_\pi W+N(U,\pi)),\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\rangle-\langle C\begin{pmatrix} U\\ \bar{U}\end{pmatrix},\dot{\pi}\partial_\pi W+N(U,\pi)\rangle\end{aligned}$$

The last four terms here shall be fairly straightforward to control. However, the first two appear to lead to a loss, as they aren't absolutely integrable. Observe that we can rewrite the sum of the first two terms as a commutator

$$\langle[\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2\nu^2\tilde{\phi}_0^4e^{2i(\Psi-\Psi_\infty)}\\ 2\nu^2\tilde{\phi}_0^4e^{-2i(\Psi-\Psi_\infty)} & 3\nu^2\tilde{\phi}_0^4\end{pmatrix},\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}C]\begin{pmatrix} U\\ \bar{U}\end{pmatrix},\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\rangle$$

The trick here is to introduce a correction function

$$t\to\theta(t):=t^2\langle\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2\nu^2\tilde{\phi}_0^4e^{2i(\Psi-\Psi_\infty)}\\ 2\nu^2\tilde{\phi}_0^4e^{-2i(\Psi-\Psi_\infty)} & 3\nu^2\tilde{\phi}_0^4\end{pmatrix}\begin{pmatrix} U\\ \bar{U}\end{pmatrix},\begin{pmatrix} U\\ \bar{U}\end{pmatrix}\rangle$$

If one applies the time derivative to $\theta(t)$, the main contribution comes from the terms when $\begin{pmatrix} U\\ \bar{U}\end{pmatrix}$ gets hit. Otherwise, one obtain at least an extra $\partial_t[\Psi-\Psi_\infty]$, which makes the expression absolutely integrable. Thus

[61]Thus by abuse of notation we use the same symbol C for this matrix-valued pseudo-conformal operator as for the scalar operator $C=x-2tp$. This shouldn't cause confusion.

$\theta'(t)$ equals up to negligible errors

$$
\begin{aligned}
i\theta'(t) &\sim t^2\langle\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)(t,y)}\nu^2\tilde{\phi}_0^4 \\ 2e^{-2i(\Psi-\Psi_\infty)(t,y)}\nu^2\tilde{\phi}_0^4 & 3\nu^2\tilde{\phi}_0^4 \end{pmatrix} i\partial_t\begin{pmatrix} U \\ \bar{U}\end{pmatrix}, \begin{pmatrix} U \\ \bar{U}\end{pmatrix}\rangle \\
&- t^2\langle\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)(t,y)}\nu^2\tilde{\phi}_0^4 \\ 2e^{-2i(\Psi-\Psi_\infty)(t,y)}\nu^2\tilde{\phi}_0^4 & 3\nu^2\tilde{\phi}_0^4 \end{pmatrix}\begin{pmatrix} U \\ \bar{U}\end{pmatrix}, i\partial_t\begin{pmatrix} U \\ \bar{U}\end{pmatrix}\rangle \\
&= t^2\langle\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)(t,y)}\nu^2\tilde{\phi}_0^4 \\ 2e^{-2i(\Psi-\Psi_\infty)(t,y)}\nu^2\tilde{\phi}_0^4 & 3\nu^2\tilde{\phi}_0^4 \end{pmatrix} \\
&\qquad [-\mathcal{H}_0\begin{pmatrix} U \\ \bar{U}\end{pmatrix} - \begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 \\ -2e^{-2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 & -3\nu^2\tilde{\phi}_0^4 \end{pmatrix}\begin{pmatrix} U \\ \bar{U}\end{pmatrix} + \ldots, \begin{pmatrix} U \\ \bar{U}\end{pmatrix}\rangle \\
&- t^2\langle\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)(t,y)}\nu^2\tilde{\phi}_0^4 \\ 2e^{-2i(\Psi-\Psi_\infty)(t,y)}\nu^2\tilde{\phi}_0^4 & 3\nu^2\tilde{\phi}_0^4 \end{pmatrix}\begin{pmatrix} U \\ \bar{U}\end{pmatrix}, \\
&\qquad [-\mathcal{H}_0\begin{pmatrix} U \\ \bar{U}\end{pmatrix} - \begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 \\ -2e^{-2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 & -3\nu^2\tilde{\phi}_0^4 \end{pmatrix}\begin{pmatrix} U \\ \bar{U}\end{pmatrix} + \ldots, \begin{pmatrix} U \\ \bar{U}\end{pmatrix}\rangle
\end{aligned}
$$

We shall see below that the terms denoted $\ldots$ lead to absolutely integrable expressions. Observe the matrix identity

$$
\begin{aligned}
&\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 \\ 2e^{-2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 & 3\nu^2\tilde{\phi}_0^4 \end{pmatrix}\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 \\ -2e^{-2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 & -3\nu^2\tilde{\phi}_0^4 \end{pmatrix} \\
&= \begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & -2e^{2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 \\ 2e^{-2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 & -3\nu^2\tilde{\phi}_0^4 \end{pmatrix}\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 \\ 2e^{-2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 & 3\nu^2\tilde{\phi}_0^4 \end{pmatrix}
\end{aligned}
$$

Thus the only contribution, up to smaller error terms, comes from the commutator with $\mathcal{H}_0$. Observe that

$$
\begin{aligned}
&t^2[\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 \\ 2e^{-2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 & 3\nu^2\tilde{\phi}_0^4 \end{pmatrix}, \mathcal{H}_0] \\
&= [\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 \\ 2e^{-2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 & 3\nu^2\tilde{\phi}_0^4 \end{pmatrix}, \tilde{C}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}] \\
&+ t^2[\begin{pmatrix} 3\nu^2\tilde{\phi}_0^4 & 2e^{2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 \\ 2e^{-2i(\Psi-\Psi_\infty)}\nu^2\tilde{\phi}_0^4 & 3\nu^2\tilde{\phi}_0^4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}],
\end{aligned}
$$

where we denote

$$
\tilde{C} = \begin{pmatrix} t^2\partial_y^2 & 0 \\ 0 & t^2\partial_y^2 \end{pmatrix}
$$

Up to an error of order $\langle(\Psi-\Psi_\infty)_2 U^2, \phi\rangle$ and hence integrable, the last term in the preceding expansion leads to the expression

$$
t^2\langle\tilde{U}^2(t,.) - \overline{\tilde{U}^2(t,.)}, \phi\rangle, \tag{5.11}
$$

for a suitable even Schwartz function[62] ϕ, where we are reverting to the notation used in the preceding section. Although this still isn't absolutely integrable, it oscillates sufficiently (due to an inherent symplectic cancellation structure) that one can integrate it over $[0,T]$. Indeed, we shall later show that for $\tilde{T} \leq T$, we have $\int_{\tilde{T}}^T t^2\langle\tilde{U}_{dis}^2(t,.) - \overline{\tilde{U}_{dis}^2(t,.)}, \phi\rangle dt \lesssim \tilde{T}^{-(\frac{1}{2}-\delta_1)}$, uniformly in T. This in addition to the bounds on the $|\lambda_i(t)| \lesssim \langle t\rangle^{-2+\delta_1}$ derived in the next subsection will suffice. We now see that up to establishing absolute integrability of the expressions

$$
t^2\langle\dot{\pi}\partial_\pi W, \begin{pmatrix} U \\ \bar{U}\end{pmatrix}\rangle,\ t^2\langle\dot{\pi}\partial_\pi W, \partial_y^i\begin{pmatrix} U \\ \bar{U}\end{pmatrix}\rangle, i=1,2,\ \langle C\begin{pmatrix} U \\ \bar{U}\end{pmatrix}, \begin{pmatrix} |U|^4U \\ -|U|^4\bar{U}\end{pmatrix}\rangle
$$

[62]This function is also time-dependent, but with uniform decay estimates in time.

plus similar contributions from the other local terms in $N(U,\pi)$, we have that

$$\int_0^T i\partial_t[\langle C\begin{pmatrix} U\\ \bar U\end{pmatrix},\begin{pmatrix} U\\ \bar U\end{pmatrix}\rangle-\frac{\theta(t)}{4}]=O(1)$$

This clearly allows us to retrieve pseudo-conformal almost-conservation. Consider $t^2\langle\dot\pi\partial_\pi W,\begin{pmatrix} U\\ \bar U\end{pmatrix}\rangle$. Using (3.17), we see that these are all equivalent to expressions of the form

$$t^2(\nu-1)(t)\langle U^2,\phi\rangle,\ t^2\lambda_6^2(t),\ t^2\lambda_6(t)\langle U,\phi\rangle$$

as well as terms of higher order in U, λ_6. These are all easily seen to be integrable in light of (3.32), (3.47), (3.48). One has to argue a bit differently when the derivative ∂_y in C falls on U, since we haven't built in a strong local dispersive estimate for ∇U. In this case, we need to leak a little extra: for a Schwartz function ϕ write

$$\phi\partial_y(U(s,.))=\phi P_{<s^{\epsilon(\frac{1}{N})}}\partial_y(U(s,.))+\phi P_{\geq s^{\epsilon(\frac{1}{N})}}\partial_y(U(s,.))$$

Note that if we choose $\epsilon(\frac{1}{N})$ large enough, we can ensure that

$$||\phi P_{\geq s^{\epsilon(\frac{1}{N})}}\partial_y(U(s,.))||_{L_x^1}\lesssim\langle s\rangle^{-N_0}$$

for large $N_0=N_0(N)$. Next, use a compactly supported partition of unity $\{\phi_j\}$ with $\phi_0(y)$ centered at $y=0$ to write

$$\phi P_{<s^{\epsilon(\frac{1}{N})}}\partial_y(U(s,.))=\sum_j\phi P_{<s^{\epsilon(\frac{1}{N})}}\partial_y(\phi_jU(s,.))$$

Then we have

$$||\phi P_{<s^{\epsilon(\frac{1}{N})}}\partial_y(\phi_jU(s,.))||_{L_x^1}\lesssim j^{-\tilde N}$$

for any $\tilde N$ and $j>s^{\epsilon_1(\frac{1}{N})}$, whence we may restrict to $j\leq s^{\epsilon_1(\frac{1}{N})}$. In that case, use the fact that the proof of the pseudo-conformal conservation law only required control of finitely many weighted estimates involving ϕ and its derivatives to conclude that

$$||\phi P_{<s^{\epsilon(\frac{1}{N})}}\partial_y(\phi_jU(s,.))||_{L_x^1}\lesssim s^{\epsilon_2(\frac{1}{N})}s^{-\frac{3}{2}}$$

Thus one obtains in summary (with a similar estimate for the 2nd derivative)

$$||\phi\partial_yU(s,.)||_{L_x^1}\lesssim\langle s\rangle^{-\frac{3}{2}+\epsilon(\frac{1}{N})}$$

Then one can proceed as before to estimate $t^2\langle\dot\pi\partial_\pi W,\partial_y^{1,2}U\rangle$. The remaining local terms in the nonlinearity shall be treatable along analogous lines, hence we now turn to the contribution of the non-local term, which is

$$\begin{aligned}&\langle C\begin{pmatrix}|U|^4U\\ -|U|^4\bar U\end{pmatrix}(t,.),\begin{pmatrix}U\\ \bar U\end{pmatrix}(t,.)\rangle\\ &=\langle\begin{pmatrix}x-2tp&0\\0&x+2tp\end{pmatrix}\begin{pmatrix}|U|^4U\\ -|U|^4\bar U\end{pmatrix}(t,.),\begin{pmatrix}x-2tp&0\\0&x+2tp\end{pmatrix}\begin{pmatrix}U\\ \bar U\end{pmatrix}(t,.)\rangle\end{aligned}\tag{5.12}$$

We have

$$(x-2tp)[|U|^4U]=-2tp[|U|^4]U+|U|^4(x-2tp)U$$

Also, we have

$$2tp[|U|^4]=4tp[|U|^2]|U|^2=2[-(x-2tp)U\bar U+U\overline{(x-2tp)U}]|U|^2$$

Thus we can expand (5.12) as a sum of expressions of the form

$$[(x-2tp)U][(x+2tp)\bar U]|U|(t)^2U^2(t)$$

plus similar terms. We can bound the L_x^1-norm of this by $\lesssim\langle t\rangle^{-2+\epsilon(\delta_2)}$, more than what we need. We note that one may similarly deduce a global bound for $||C\partial_yU(t,.)||_{L_x^2}$. But we only need the bound $||C\partial_yU||_{L_x^2}\lesssim\langle t\rangle^{\frac{1}{2}}$, anyways, see the last subsection. We are now done with establishing the estimates for U_{dis}, up to bounding (5.11), which we shall do later.

5.3. **Retrieving control over the root part.** We now improve the bound on

$$\sup_{0\leq t<T}\sup_{0\leq k\leq[\frac{N}{3}]}||\langle t\rangle^{2-4\delta_1}\frac{d^k}{dt^k}\lambda_i(t)||_{L^M} \tag{5.13}$$

First, note from (3.34), (3.35) as well as (3.24), (3.25) that we may immediately achieve this if we have improved the estimates for $\lambda_6(t)$. Again we shall suppress Λ in the following, it being clear that sufficiently many δ's will come up to improve the bound. Now recall the ODE (3.40), as well as (3.41). We brutally expand $e^{\Lambda(s)}$ into a Taylor series. Then note that the worst[63] contribution in [...] in (3.41) comes from the terms $(\nu-1)^a(s)\langle\begin{pmatrix}\tilde{U}\\ \overline{\tilde{U}}\end{pmatrix}_{dis},\phi\rangle(s)$, $a=1,2$. Indeed, all other expressions are easily seen to contribute a term decaying like $\langle s\rangle^{-3+4\delta_1}$ or faster. In order to treat these bad terms, we have to recycle the equation again(as we may). Employing Duhamel expansion as usual, we have to bound the following terms (in the case $a=1$ e. g.)

$$\int_t^\infty(\nu-1)(s)\langle\int_0^s e^{i(s-\lambda)\mathcal{H}}[\begin{pmatrix}0 & -e^{-2i(\Psi-\Psi_\infty)_1(s)+2i(\Psi-\Psi_\infty)_1(\lambda)}+1\\ e^{2i(\Psi-\Psi_\infty)_1(s)-2i(\Psi-\Psi_\infty)_1(\lambda)}-1 & \end{pmatrix}\begin{pmatrix}\tilde{U}^{(s)}(\lambda,.)\\ \overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}\phi]_{dis},\phi\rangle d\lambda ds \tag{5.14}$$

$$\int_t^\infty(\nu-1)(s)\langle\int_0^s e^{i(s-\lambda)\mathcal{H}}\begin{pmatrix}|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.)\\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}_{dis},\phi\rangle d\lambda ds \tag{5.15}$$

We commence with the first of these: introduce the matrix valued function

$$\phi(s,\lambda,x):=\begin{pmatrix}0 & -e^{-2i(\Psi-\Psi_\infty)_1(s)+2i(\Psi-\Psi_\infty)_1(\lambda)}+1\\ e^{2i(\Psi-\Psi_\infty)_1(s)-2i(\Psi-\Psi_\infty)_1(\lambda)}-1 & \end{pmatrix}\phi(x)$$

Using the Plancherel's Theorem 2.3 for the distorted Fourier transform, we can express the first term as

$$\sum_\pm\int_{-\infty}^\infty\int_t^\infty(\nu-1)(s)\int_0^s e^{\pm i(s-\lambda)(\xi^2+1)}\mathcal{F}_\pm[\phi(s,\lambda,x)\begin{pmatrix}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{pmatrix}(\lambda,.)](\xi)\overline{\tilde{\mathcal{F}}_\pm\phi(\xi)}d\lambda ds d\xi$$

We perform an integration by parts in the s-variable, replacing the above by the following terms:

$$\sum_\pm\int_{-\infty}^\infty(\nu-1)(t)\int_0^t e^{\pm i(t-\lambda)(\xi^2+1)}\mathcal{F}_\pm[\phi(t,\lambda,x)\begin{pmatrix}\tilde{U}^{(t)}\\ \overline{\tilde{U}^{(t)}}\end{pmatrix}(\lambda)](\xi)\frac{\overline{\tilde{\mathcal{F}}_\pm\phi(\xi)}}{\xi^2+1}d\lambda d\xi \tag{5.16}$$

$$\sum_\pm\int_{-\infty}^\infty\int_t^\infty(\nu-1)(s)\partial_s[\Psi-\Psi_\infty]_1(s)\int_0^s e^{\pm i(s-\lambda)(\xi^2+1)}\mathcal{F}_\pm[\tilde{\phi}(s,\lambda,x)\begin{pmatrix}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{pmatrix}](\xi)\frac{\overline{\tilde{\mathcal{F}}_\pm\phi(\xi)}}{\xi^2+1}d\lambda ds d\xi \tag{5.17}$$

$$\sum_\pm\int_{-\infty}^\infty\int_t^\infty(\nu-1)(s)\partial_s[\lambda_\infty(\mu-\mu_\infty)](s)\int_0^s e^{\pm i(s-\lambda)(\xi^2+1)}\mathcal{F}_\pm[\phi(s,\lambda,x)\partial_y\begin{pmatrix}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{pmatrix}](\xi)\frac{\overline{\tilde{\mathcal{F}}_\pm\phi(\xi)}}{\xi^2+1}d\lambda ds d\xi \tag{5.18}$$

$$\sum_\pm\int_{-\infty}^\infty\int_t^\infty\dot{\nu}(s)\int_0^s e^{\pm i(s-\lambda)(\xi^2+1)}\mathcal{F}_\pm[\phi(s,\lambda,x)\begin{pmatrix}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{pmatrix}(\lambda,.)](\xi)\frac{\overline{\tilde{\mathcal{F}}_\pm\phi(\xi)}}{\xi^2+1}d\lambda ds d\xi \tag{5.19}$$

[63] This is of course a naive qualification; this term actually oscillates a lot.

Most of these are almost immediate to estimate. Note that upon undoing the Fourier transform in the first expression, we obtain

$$(\nu-1)(t)\int_0^t \langle e^{i(t-\lambda)\mathcal{H}}[\left(\begin{array}{c}\tilde{U}(\lambda,.)\\ \tilde{\bar{U}}(\lambda,.)\end{array}\right)\phi(t,\lambda,x)]_{dis},(\mathcal{H}^*)^{-1}\phi(x)\rangle d\lambda$$

We claim that $||x^2\mathcal{H}^{-1}\phi||_{L^{1+}_x}\lesssim O(1)$, where 1+ can be chosen arbitrarily close to 1. Write

$$(\mathcal{H})^{-1}\phi=\sum_{\pm}\int_{-\infty}^{\infty}e_{\pm}(x,\xi)\frac{\tilde{\mathcal{F}}_{\pm}(\phi)(\xi)}{\xi^2+1}d\xi \tag{5.20}$$

We treat here the + part, the other one being similar. We equate the preceding for $x>0$ with

$$\int_0^{\infty}[s(\xi)e^{ix\xi}\underline{e}+\phi(x,\xi)]\frac{\mathcal{F}_{\pm}(\phi)(\xi)}{\xi^2+1}d\xi+\int_{-\infty}^{0}([e^{ix\xi}+r(-\xi)e^{-ix\xi}]\underline{e}+\phi(x,\xi))\frac{\mathcal{F}(\phi)(\xi)}{\xi^2+1}d\xi$$

For the first integral, leaving out the trivial local part, we have

$$\int_0^{\infty}s(\xi)e^{ix\xi}\underline{e}\frac{\mathcal{F}_{\pm}(\phi)(\xi)}{\xi^2+1}d\xi=-\frac{1}{ix}\int_0^{\infty}(\partial_\xi[s(\xi)]e^{ix\xi}\frac{\mathcal{F}_{\pm}(\phi)(\xi)}{\xi^2+1}\underline{e}+s(\xi)e^{ix\xi}\underline{e}\partial_\xi[\frac{\mathcal{F}_{\pm}(\phi)(\xi)}{\xi^2+1}])d\xi$$
$$=+\frac{1}{-x^2}\int_0^{\infty}e^{ix\xi}\partial_\xi(\partial_\xi[s(\xi)]\frac{\mathcal{F}_{\pm}(\phi)(\xi)}{\xi^2+1}\underline{e})d\xi+\frac{1}{-x^2}\int_0^{\infty}e^{ix\xi}\partial_\xi[s(\xi)\underline{e}\partial_\xi[\frac{\mathcal{F}_{\pm}(\phi)(\xi)}{\xi^2+1}]]d\xi=O(\frac{1}{x^3})$$

Similarly (omitting the contributions from $1+r(-\xi)$, $\phi(x,\xi)$), we have

$$\int_{-\infty}^{0}\frac{e^{ix\xi}-e^{-ix\xi}}{\xi^2+1}\underline{e}\mathcal{F}_{\pm}(\phi)(\xi)d\xi=-\frac{1}{ix}\int_{-\infty}^{0}[e^{ix\xi}+e^{-ix\xi}]\underline{e}\partial_\xi[\frac{\mathcal{F}_{\pm}\phi(\xi)}{\xi^2+1}]d\xi$$
$$=+\frac{1}{-x^2}\int_{-\infty}^{0}(e^{ix\xi}-e^{-ix\xi})\underline{e}\partial_\xi^2[\frac{\mathcal{F}_{\pm}\phi(\xi)}{\xi^2+1}]d\xi=O(\frac{1}{x^3})$$

If we then repeat the steps in the proof of the strong local dispersive estimate, we get

$$|(\nu-1)(t)\int_0^t\langle e^{\pm i(t-\lambda)\mathcal{H}}[\left(\begin{array}{c}\tilde{U}(\lambda,.)\\ \tilde{\bar{U}}(\lambda,.)\end{array}\right)\phi(t,\lambda,x)]_{dis},(\mathcal{H}^*)^{-1}\phi(x)\rangle d\lambda|\lesssim t^{-\frac{1}{2}+\delta_1}t^{-\frac{3}{2}+2\delta_3},$$

as desired. In the expression (5.17), carry out two additional integrations by parts. This either produces additional factors of at least the decay $\nu-1$, or else kills the integral over λ, in which case one arrives at an expression[64]

$$\int_t^{\infty}(\nu-1)(s)\frac{d}{ds}[\Psi-\Psi_{\infty}]_1(s)\langle\left(\begin{array}{c}\overline{\tilde{U}}\\ \tilde{\bar{U}}\end{array}\right)(s,.)\phi(x),(\mathcal{H}^*)^{-2}\phi_{dis}\rangle ds$$

Then note that $\langle\left(\begin{array}{c}\overline{\tilde{U}}\\ \tilde{\bar{U}}\end{array}\right)(s,.)\phi(x),(\mathcal{H}^*)^{-2}\phi_{dis}\rangle=\overline{\langle\left(\begin{array}{c}\tilde{U}\\ \tilde{\bar{U}}\end{array}\right)(s,.)\phi(x),(\mathcal{H}^*)^{-2}\overline{\phi_{dis}}\rangle}$. Now use the customary decomposition of $\left(\begin{array}{c}\tilde{U}\\ \tilde{\bar{U}}\end{array}\right)$ into dispersive and root part. Recycling (3.35), we thus see that up to an integral of the form

$$\int_t^{\infty}(\nu-1)^2(t)\lambda_6(t)dt, \tag{5.21}$$

we arrive at the expression we started out with but with an extra weight of at least the strength $(\nu-1)(s)\frac{d}{ds}[\Psi-\Psi_{\infty}]_1(s)\sim(\nu(s)-1)^2$. Now iterate the procedure. The integral (5.21) can be estimated using Proposition 5.4. The remaining terms are simpler; indeed, they can be integrated absolutely to give the desired upper bound. Now consider (5.15). Here we also pass to the Fourier side, perform an integration by parts in s, and undo

[64]Using the distorted Plancherel's theorem 2.3.

the Fourier transform. This results in the extra weights $\partial_s[\Psi-\Psi_\infty]_1$, $\partial_s[\lambda_\infty(\mu-\mu_\infty)]$, or else one winds up with the expression

$$\int_t^\infty \langle \left(\begin{array}{c} |\tilde{U}|^4\tilde{U}(s) \\ -|\tilde{U}|^4\tilde{U}(s) \end{array}\right)_{dis}, \mathcal{H}^{-1}\phi\rangle ds$$

The former cases are treated by repeating integration by parts in s if necessary and proceeding as in the proof of the strong local dispersive estimate with ϕ replaced by $\mathcal{H}^{-1}\phi$, while in the latter case we simply estimate (using the bound derived above on $||\langle x\rangle^2\mathcal{H}^{-1}\phi||_{L_x^{1+}}$)

$$|\int_t^\infty \langle \left(\begin{array}{c} |\tilde{U}|^4\tilde{U}(s) \\ -|\tilde{U}|^4\tilde{U}(s) \end{array}\right)_{dis}, \mathcal{H}^{-1}\phi\rangle ds| \lesssim \int_t^\infty s^{-\frac{9}{2+\epsilon(\delta_2)}} ds \lesssim t^{-2+2\delta_1}$$

Finally, obtaining estimates for $\frac{d^k}{dt^k}\lambda_6$, $1\leq k\leq [\frac{N}{3}]$, is straightforward upon differentiating (3.40). We have used a sleight of hand here, since suppressed the possible time dependence of ϕ in $(\nu-1)\langle \left(\begin{array}{c} \tilde{U} \\ \overline{\tilde{U}} \end{array}\right)_{dis}, \phi\rangle$. However, as already mentioned in the derivation of the equation for λ_6, the derivative of ϕ with respect to time has at least the decay of $\dot{\nu}$. Inspecting the above proof, one easily checks that the additional terms generated upon integration by parts can be handled by the same method. We are now done with the a priori estimates for U.

5.4. **Interlude: deriving a refined estimate for $\lambda_6(t)$.** This is the most challenging subsection, and condenses all the preceding considerations into one crucial estimate for $\lambda_6(t)$ which appears indispensable to close the estimates for the modulation parameters. Indeed, all our travails in establishing the refined local decay estimates for U are really leading up to and flowing into this estimate, which we state as the following Proposition:

Proposition 5.4. *Let $\Gamma\in A^{(n)}[0,T)$, $0<T\leq\infty$, see Theorem 4.5. Then for $\tilde{T}\leq T$ and*

$$\Gamma=\{\left(\begin{array}{c} \tilde{U} \\ \overline{U} \end{array}\right)_{dis},\dots,\lambda_6(t),\dots\}$$

we have the bound

$$|\int_{\tilde{T}}^T t\lambda_6(t)dt| \lesssim \delta^2\langle\tilde{T}\rangle^{-\frac{1}{2}+\delta_1}$$

uniformly in T. Also, we have

$$|\int_{\tilde{T}}^T (\nu-1)^a\lambda_6(t)dt| \lesssim \delta^2\langle\tilde{T}\rangle^{-\frac{3}{2}+\delta_1+a(-\frac{1}{2}+\delta_1)},\ a\geq 0$$

Proof. This estimate is clearly significantly more difficult than what we established in the previous subsection[65]. We may put $T=\infty$, the more general case being treated identically. Also, we shall prove the first inequality, the 2nd following from the same proof. We shall again recycle (3.41), which amongst other expressions will lead to $\int_T^\infty t\int_t^\infty\langle\tilde{U}^2(s,.)-\overline{\tilde{U}}^2(s,.),\phi\rangle dsdt$. The treatment of the latter shall also be applicable to $\int_T^\infty t^2\langle\tilde{U}^2(t,.)-\overline{\tilde{U}}^2(t,.),\phi\rangle dt$, which will fill in the hole in retrieving the bound for $\sup_{0\leq t<T}||CU(t,.)||_{L_x^2}$. The logic of the argument below shall be that if an expression can't be integrated absolutely, we integrate by parts until either it can be integrated absolutely, or else we wind up essentially in the position we started out with but with an extra gain. Thus in the proof of the Lemmata below, it may be that we arrive at terms just as the first one in Proposition 5.4, but with $t\lambda_6(t)$ replaced by $(\nu(t)-1)t\lambda_6(t)$. The idea then is to reiterate the whole process again. Thus the Lemmata below should be thought of as being proved in tandem. Now from (3.40) we need to estimate a number of expressions, the first of which is

$$\int_T^\infty t\int_t^\infty(\nu-1)(s)\langle\left(\begin{array}{c} \tilde{U} \\ \overline{\tilde{U}} \end{array}\right)_{dis},\phi\rangle ds$$

[65]We did use the last conclusion of the Proposition to deduce the point wise estimates for λ_6, but this was certainly an overkill.

Lemma 5.5. *The following inequality holds under the assumptions of Proposition 5.4:*

$$|\int_T^\infty t\int_t^\infty (\nu-1)(s)\langle \begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}(s,.),\phi\rangle ds| \lesssim \delta^3\langle T\rangle^{-\frac{1}{2}+\delta_1}$$

Proof. We Duhamel-expand $\begin{pmatrix} \tilde{U} \\ \bar{\tilde{U}} \end{pmatrix}_{dis}$. This leads to the expressions

$$\int_T^\infty t\int_t^\infty(\nu-1)(s)\langle\int_0^s e^{i(s-\lambda)\mathcal{H}}[\begin{pmatrix} (1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)})\overline{\tilde{U}^{(s)}}(\lambda) \\ (-1+e^{-2i(\Psi-\Psi_\infty)_1(\lambda)+2i(\Psi-\Psi_\infty)_1(s)})\tilde{U}^{(s)}(\lambda)\end{pmatrix}\phi]_{dis}d\lambda,\phi\rangle dsdt \tag{5.22}$$

$$\int_T^\infty t\int_t^\infty(\nu-1)(s)\langle\int_0^s e^{i(s-\lambda)\mathcal{H}}\begin{pmatrix} |\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -|\tilde{U}^{(s)}|^4\bar{\tilde{U}}^{(s)}(\lambda,.)\end{pmatrix}_{dis}d\lambda,\phi\rangle dsdt \tag{5.23}$$

as well as local terms with better decay behavior than the first expression; these can be treated analogously. Start with the first expression. We perform an integration by parts in s, and replace it by the following list of terms upon going to the Fourier side. We leave out $\pm$ for simplicity:

(5.24)
$$\int_{-\infty}^\infty\int_T^\infty t(\nu-1)(t)\int_0^t e^{i(t-\lambda)(\xi^2+1)}\mathcal{F}[\begin{pmatrix} (1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(t)})\overline{\tilde{U}^{(t)}}(\lambda,.) \\ (-1+e^{-2i(\Psi-\Psi_\infty)_1(\lambda)+2i(\Psi-\Psi_\infty)_1(t)})\tilde{U}^{(t)}(\lambda,.)\end{pmatrix}\phi](\xi)\overline{\frac{\tilde{\mathcal{F}}\phi}{\xi^2+1}}d\lambda dtd\xi$$

(5.25)
$$\int_{-\infty}^\infty\int_T^\infty t\int_t^\infty \dot{\nu}(s)\int_0^s e^{i(s-\lambda)(\xi^2+1)}\mathcal{F}[\begin{pmatrix} (1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)})\overline{\tilde{U}^{(s)}}(\lambda,.) \\ (-1+e^{-2i(\Psi-\Psi_\infty)_1(\lambda)+2i(\Psi-\Psi_\infty)_1(s)})\tilde{U}^{(s)}(\lambda,.)\end{pmatrix}\phi](\xi)\overline{\frac{\tilde{\mathcal{F}}\phi}{\xi^2+1}}d\lambda dtd\xi$$

$$\begin{aligned}&\int_{-\infty}^\infty\int_T^\infty t\int_t^\infty(\nu-1)(s)\frac{d}{ds}[\Psi-\Psi_\infty]_1\\ &\quad\int_0^s e^{i(s-\lambda)(\xi^2+1)}\mathcal{F}[\begin{pmatrix} (1+e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)})\overline{\tilde{U}^{(s)}}(\lambda,.) \\ (1+e^{-2i(\Psi-\Psi_\infty)_1(\lambda)+2i(\Psi-\Psi_\infty)_1(s)})\tilde{U}^{(s)}(\lambda,.)\end{pmatrix}\phi](\xi)\overline{\frac{\tilde{\mathcal{F}}\phi}{\xi^2+1}}d\lambda dtd\xi\end{aligned} \tag{5.26}$$

(5.27)
$$\begin{aligned}&\int_{-\infty}^\infty\int_T^\infty t\int_t^\infty(\nu-1)(s)\frac{d}{ds}[\lambda_\infty(\mu-\mu_\infty)]\\ &\quad\int_0^s e^{i(s-\lambda)(\xi^2+1)}\mathcal{F}[\begin{pmatrix} (1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)})\overline{\partial_x\tilde{U}^{(s)}}(\lambda,.) \\ (-1+e^{-2i(\Psi-\Psi_\infty)_1(\lambda)+2i(\Psi-\Psi_\infty)_1(s)})\partial_x\tilde{U}^{(s)}(\lambda,.)\end{pmatrix}\phi](\xi)\overline{\frac{\tilde{\mathcal{F}}\phi}{\xi^2+1}}d\lambda dtd\xi\end{aligned}$$

A similar list of terms results from (5.23). We commence with the first term in this list. Perform an integration by parts in t, thereby replacing it by

$$\int_{-\infty}^\infty T(\nu-1)(T)\int_0^T e^{i(T-\lambda)(\xi^2+1)}\mathcal{F}[\begin{pmatrix} (1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(T)})\overline{\tilde{U}^{(T)}}(\lambda,.) \\ (-1+e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(T)})\tilde{U}^{(T)}(\lambda,.)\end{pmatrix}\phi](\xi)\overline{\frac{\tilde{\mathcal{F}}\phi}{(\xi^2+1)^2}}d\lambda dTd\xi$$

$$\int_{-\infty}^\infty\int_T^\infty(\nu-1)(t)\int_0^t e^{i(t-\lambda)(\xi^2+1)}\mathcal{F}[\begin{pmatrix} (1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(t)})\overline{\tilde{U}^{(t)}}(\lambda,.) \\ (-1+e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(t)})\tilde{U}^{(t)}(\lambda,.)\end{pmatrix}\phi](\xi)\overline{\frac{\tilde{\mathcal{F}}\phi}{(\xi^2+1)^2}}d\lambda dtd\xi$$

$$\begin{aligned}&\int_{-\infty}^\infty\int_T^\infty t(\nu-1)(t)\frac{d}{dt}[\Psi-\Psi_\infty]_1(t)\\ &\quad\int_0^t e^{i(t-\lambda)(\xi^2+1)}\mathcal{F}[\begin{pmatrix} (1+e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(t)})\overline{\tilde{U}^{(t)}}(\lambda,.) \\ (1+e^{-2i(\Psi-\Psi_\infty)(\lambda)+2i(\Psi-\Psi_\infty)(t)})\tilde{U}^{(t)}(\lambda,.)\end{pmatrix}\phi](\xi)\overline{\frac{\tilde{\mathcal{F}}\phi}{(\xi^2+1)^2}}d\lambda dtd\xi\end{aligned}$$

$$\int_{-\infty}^{\infty}\int_{T}^{\infty} t(\nu-1)(t)\frac{d}{dt}[\lambda_{\infty}(\mu-\mu_{\infty})](t)$$
$$\int_{0}^{t} e^{i(t-\lambda)(\xi^2+1)}\mathcal{F}[\left(\begin{array}{c}(1-e^{2i(\Psi-\Psi_{\infty})_1(\lambda)-2i(\Psi-\Psi_{\infty})_1(t)})\overline{\partial_x\tilde{U}^{(t)}}(\lambda,.)\\ (-1+e^{-2i(\Psi-\Psi_{\infty})_1(\lambda)+2i(\Psi-\Psi_{\infty})_1(t)})\partial_x\tilde{U}^{(t)}(\lambda,.)\end{array}\right)\phi](\xi)\overline{\frac{\tilde{\mathcal{F}}\phi}{(\xi^2+1)^2}}d\lambda dt d\xi$$

Each of these terms is straightforward to estimate: undo the Fourier transform, using the distorted Plancherel's Theorem 2.3, thereby replacing ϕ by $\mathcal{H}^{-2}\phi$, which again satisfies $||x^2\mathcal{H}^{-2}\phi||_{L_x^{1+}}\lesssim O(1)$. Thus, for example we get

$$|\int_{-\infty}^{\infty} T(\nu-1)(T)\int_{0}^{T} e^{i(T-\lambda)(\xi^2+1)}\mathcal{F}[\left(\begin{array}{c}(1-e^{2i(\Psi-\Psi_{\infty})_1(\lambda)-2i(\Psi-\Psi_{\infty})_1(T)})\overline{\tilde{U}^{(T)}}(\lambda)\\ (-1+e^{-2i(\Psi-\Psi_{\infty})_1(\lambda)+2i(\Psi-\Psi_{\infty})_1(T)})\tilde{U}^{(T)}(\lambda)\end{array}\right)\phi](\xi)\overline{\frac{\tilde{\mathcal{F}}\phi}{(\xi^2+1)^2}}d\lambda dT d\xi|$$
$$\lesssim T^{\frac{1}{2}+\delta_1}\langle T\rangle^{-\frac{3}{2}}dT\lesssim T^{-1+\delta_1},$$

better than what we need. The 2nd and 4th term in the immediately preceding list are estimated similarly. For the third term, perform an additional integration by parts. Either one pulls down an additional factor of at least the decay $\frac{d}{dt}[\Psi-\Psi_{\infty}]_1(t)$, in which case one can integrate absolutely to get the desired upper bound, or[66] one obtains expressions of the form[67]

$$\int_{T}^{\infty} t(\nu(t)-1)^{2+a}\langle\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis}(t,.),\phi\rangle dt,\ a=0,1$$

One can easily handle this by further Duhamel expansion, which we leave out. We now turn to (5.25). This is more difficult, as the SLDE[68] together with (3.32) do not suffice to obtain the needed $\langle T\rangle^{-\frac{1}{2}+\delta_1}$ decay. Thus we need to exploit that we may reiterate the equation, and use (3.24), which implies

$$\begin{aligned}\dot{\nu}(s)=&-b_{\infty}\int_{s}^{\infty}\lambda_{\infty}^{-1}(\sigma)[\nu(\sigma)(4i\kappa_2)^{-1}E_5(\sigma)+\beta\nu(\nu-1)^2(\sigma)+(2\nu-1)(\beta\nu-b_{\infty}\lambda_{\infty}^{-1}(\sigma))]d\sigma\\ &+\nu(s)(4i\kappa_2)^{-1}E_5(s)+\beta\nu(\nu-1)^2(s)+(2\nu-1)(\beta\nu-b_{\infty}\lambda_{\infty}^{-1})(s)\end{aligned}$$

If we substitute the first line here for $\dot{\nu}(s)$ into (5.25), we perform further integrations by parts in s, which either produces produces something which we can integrate absolutely to get the desired upper bound (namely when we differentiate the integral expression in the first row, or when we pick up at least two factors $\frac{d}{ds}[\Psi-\Psi_{\infty}]_1$), or else we arrive at an expression just as in the statement of the Lemma but with an extra weight decaying like $\dot{\nu}$, in which case we reiterate the whole process. Also, substituting $\beta\nu(\nu-1)^2$ instead of $\dot{\nu}$ is easily seen to yield more than what is required upon performing further integrations by parts. The real difficulty comes from substituting $\nu(4i\kappa_2)^{-1}E_5(s)$, since a priori it isn't clear whether the derivative of this term decays faster than $\langle s\rangle^{-2+\delta_1+\delta_3}$. Now recall[69] that

$$E_5=-\langle N,\tilde{\xi}_5\rangle+\langle U,(i\partial_s+\mathcal{H}^*(s))\tilde{\xi}_5\rangle$$

The first term on the right decays at least as fast as $\langle s\rangle^{-3+2\delta_3}$, and is easily seen to cause no problems. The 2nd term on the right is essentially of the form $(\nu-1)(s)\langle\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis},\phi\rangle$, plus errors which are again negligible. Thus, in order to control (5.25), we need to substitute $(\nu-1)(s)\langle\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis},\phi\rangle$ for $\dot{\nu}(s)$; our only hope of succeeding here is to Duhamel-expand this 2nd instance of $\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis}$. Pausing in our analysis here for a

[66] In the case when the inner integral gets abolished

[67] Use the decomposition $\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)=\left(\begin{array}{c}\tilde{U}\\ \bar{\tilde{U}}\end{array}\right)_{dis}+\sum_{i=1}^{6}\lambda_i\eta_{i,\mathrm{proper}}$; we shall soon see that neither λ_2 nor λ_6 contribute anything to the expression in question, hence the expressions, using (3.35).

[68] Strong local dispersive estimate, i. e. $||\phi U(t,.)||_{L_x^{\infty}}\lesssim\langle t\rangle^{-\frac{3}{2}+\delta_3}$.

[69] See (3.17)

moment, observe that we run into quite similar issues upon integrating by parts with respect to s in (5.23). Thus what our problem really boils down to is estimating the expression

$$\int_T^\infty t\int_t^\infty (\nu-1)(s)\langle\left(\begin{array}{c}\tilde{U}\\ \tilde{\bar{U}}\end{array}\right)_{dis}(s,.),\phi\rangle\langle\left(\begin{array}{c}\tilde{U}\\ \tilde{\bar{U}}\end{array}\right)_{dis}(s,.),(\mathcal{H}^*)^{-1}\psi\rangle ds, \tag{5.28}$$

where we use schematic notation. Let's assume for now that we can bound this expression by $\langle T\rangle^{-\frac{1}{2}+\delta_1}$. Finally, substituting $\beta\nu-\frac{b_\infty}{\lambda_\infty}$ for $\dot{\nu}$ is treated as in the preceding case: perform an additional integration by parts in s; either one produces an extra factor $\frac{d}{ds}[\Psi-\Psi_\infty]_1(s)$, in which case one reiterates integration by parts to arrive either at a much improved expression like in the statement of the Lemma, or to arrive at an expression which can be integrated absolutely to yield the bound $T^{-\frac{1}{2}+\delta_1}$. If on the other hand one differentiates $\beta\nu-\frac{b_\infty}{\lambda_\infty}$, use (3.25): The worst contribution there comes from $E_2(s)\sim\lambda_6(s)$. This one treats by another integration by parts, which either produces an extra $\frac{d}{ds}[\Psi-\Psi_\infty]_1(s)$ whence one can integrate absolutely, or else one gets $\dot{\lambda}_6(s)$, which is treated by recycling (3.40). There the only dangerous contribution comes from $(\nu-1)\langle\left(\begin{array}{c}\tilde{U}\\ \tilde{\bar{U}}\end{array}\right)_{dis},\phi\rangle$, which leads to an expression as in (5.28). Let's move on to (5.26), reiterate integration by parts in s; either one hits $(\nu-1)(s)\frac{d}{ds}[\Psi-\Psi_\infty]_1$, with $\frac{d}{ds}$, or one abolishes the integration in s, or one produces at least an extra factor $\frac{d}{ds}[\Psi-\Psi_\infty]_1$. In the first case, we obtain an expression like in (5.25) but with an extra weight $\nu-1$, whence we can treat this case just as above (actually, this time the contribution of E_5 can just be integrated absolutely). In the third case, reiterate integration by parts, which either takes one into the first two cases, or else produces an additional $\frac{d}{ds}[\Psi-\Psi_\infty]_1$. In the last case, keep integrating by parts, which either eventually produces arbitrary gains in s, or else takes one into the first two cases. The 2nd case is more tricky: Observe that then we obtain an expression of the form[70]

$$\int_T^\infty t\int_t^\infty (\nu(s)-1)\frac{d}{ds}[\Psi-\Psi_\infty]_1(s)\langle\left(\begin{array}{c}\overline{\tilde{U}}\\ \tilde{U}\end{array}\right)(s,x)\phi(x),\,(\mathcal{H}^*)^{-2}\phi_{dis}\rangle ds$$

Observe that a priori the factor $\left(\begin{array}{c}\overline{\tilde{U}}\\ \tilde{U}\end{array}\right)$ might imply the presence of a $\lambda_6(s)$, which would lead to an extremely difficult term. However, close inspection of the argument for the equation of λ_6 shows that in every expression $(\nu-1)\langle\left(\begin{array}{c}\tilde{U}\\ \tilde{\bar{U}}\end{array}\right)_{dis},\phi\rangle$ *with exactly one power* of $\nu-1$, the (vector valued) function ϕ has the form $\left(\begin{array}{c}\alpha\\ -\alpha\end{array}\right)$ and is real valued. This then implies that $\phi_{dis}=\left(\begin{array}{c}\tilde{\alpha}\\ -\tilde{\alpha}\end{array}\right)$, and then also $(\mathcal{H}^*)^{-2}\phi_{dis}=\left(\begin{array}{c}\beta\\ -\beta\end{array}\right)$, with real valued scalar function β. Thus we get $\langle\eta_{6,\mathrm{proper}}\phi(x),\left(\begin{array}{c}\beta\\ -\beta\end{array}\right)=0$, and similarly $\langle\eta_{2,\mathrm{proper}}\phi(x),\left(\begin{array}{c}\beta\\ -\beta\end{array}\right)=0$, whence

$$\langle\left(\begin{array}{c}\overline{\tilde{U}}\\ \tilde{U}\end{array}\right)(s,x)\phi(x),(\mathcal{H}^*)^{-2}\phi_{dis}\rangle=\langle\left(\begin{array}{c}\overline{\tilde{U}}\\ \tilde{U}\end{array}\right)_{dis}\phi(x),(\mathcal{H}^*)^{-2}\phi_{dis}\rangle+\sum_{i\neq 2,6}\langle\eta_{i,\mathrm{proper}}\phi(x),(\mathcal{H}^*)^{-2}\phi_{dis}\rangle$$

Thus using (3.35) we have put ourselves into basically the situation we started out with in the Lemma (up to negligible error terms), but with a weight $(\nu-1)(s)\frac{d}{ds}[\Psi-\Psi_\infty]_1\sim(\nu-1)^2(s)$. Now reiterate the whole process. Observe that performing the same reasoning for the expressions of the form

$$\int_T^\infty t\int_t^\infty (\nu(s)-1)^2\langle\left(\begin{array}{c}\tilde{U}\\ \tilde{\bar{U}}\end{array}\right)_{dis}(s,.)\phi\rangle ds dt$$

which are also implied by (3.40) we may very well arrive at a term of the form

$$\int_T^\infty t\int_t^\infty (\nu(s)-1)^2(\frac{d}{ds}[\Psi-\Psi_\infty](s))\lambda_6(s)ds$$

[70]The function $\phi(x)$ here is scalar- and real valued; indeed, one checks that $\phi(x)=\phi_0^4(x)$.

Upon integration by parts[71], this leads to

$$\int_T^\infty t\int_t^\infty (\int_s^\infty (\nu(\sigma)-1)^3 d\sigma)\dot{\lambda_6}(s)ds$$

$$\int_T^\infty t\int_t^\infty (\nu(\sigma)-1)^3 d\sigma\lambda_6(t)dt$$

The last term is what we started out with in Proposition 5.4, but with an extra factor $\int_t^\infty(\nu(\sigma-1)^3 d\sigma$. Then reiterate the whole process. For the first expression, recycle (3.40); one winds up with terms just as in the Lemma, but with the extra weight $\int_s^\infty(\nu(\sigma)-1)^3 d\sigma$. Now reiterate the process. Finally, the term (5.27) is also treated by expanding $\frac{d}{ds}[\lambda_\infty(\mu-\mu_\infty)](s)$ using (3.29). One obtains terms which can either be handled by further integrations by parts in s, or else one is led to an expression just as (5.28). As already mentioned, the expression (5.23) is treated by exact analogy. One reiterates the proof of the SLDE[72] for each instance of $\langle\int_0^t\langle e^{i(t-s)\mathcal{H}}\begin{pmatrix}|\tilde{U}|^{(t)}\tilde{U}^{(t)}\\ -|\tilde{U}|^{(t)}\overline{\tilde{U}^{(t)}}\end{pmatrix}_{dis}(s,.),(\mathcal{H}^*)^{-k}\phi\rangle ds$. This concludes the proof of the Lemma up to the assertion concerning (5.28). □

We thus need

Lemma 5.6. [73] *Under the assumptions of the previous Lemma, we have*

$$|\int_T^\infty t\int_t^\infty(\nu-1)(s)\langle\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}(s,.),\phi\rangle\langle\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}(s,.),(\mathcal{H}^*)^{-k}\psi\rangle ds|\lesssim\langle T\rangle^{-\frac{1}{2}+\delta_1} \tag{5.29}$$

for any $k\geq 0$ and Schwartz functions ϕ,ψ.

Proof. We Duhamel-expand each copy of $\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}(s,.)$:

$$\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}(s,.)=e^{is\mathcal{H}}\begin{pmatrix}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{pmatrix}_{dis}(0,.)$$

$$+\int_0^s e^{i(s-\lambda)\mathcal{H}}[\begin{pmatrix}0 & 1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)}\\ 1-e^{-2i(\Psi-\Psi_\infty)_1(\lambda)+2i(\Psi-\Psi_\infty)_1(s)} & 0\end{pmatrix}\begin{pmatrix}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{pmatrix}]_{dis}d\lambda+\dots$$

$$+\int_0^s e^{i(s-\lambda)\mathcal{H}}\begin{pmatrix}|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.)\\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}_{dis}d\lambda:=A+B+\dots+C$$

The terms $\dots$ are local terms with better decay properties and can be treated in a simpler fashion, hence left out. We then substitute either A,B or C for $\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}(s,.)$ in the right hand side of (5.29) and check that the resulting expression has the same decay as $(\nu-1)(T)$. The logic behind these estimates is as follows: in an expression of the form

$$\langle e^{i(s-\lambda)\mathcal{H}}E,\phi_{dis}\rangle\langle e^{i(s-\lambda)\mathcal{H}}F,\mathcal{H}^{-k}\psi_{dis}\rangle=\langle E,e^{-i(s-\lambda)\mathcal{H}^*}\phi_{dis}\rangle\langle F,e^{-i(s-\lambda)\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle,$$

distinguish between the case when ϕ_{dis} and ψ_{dis} have separated Fourier support, respectively closely aligned (correlated) Fourier support. In the former case, the product oscillates strongly with respect to s, whence one can integrate by parts and hope to gain. In the latter case, one should be able to exploit some kind of 'diagonalization effect' in order to gain. The argument proceeds by distinguishing between several interactions:

[71] Also, use $\frac{d}{ds}[\Psi-\Psi_\infty]_1\sim(\nu-1)(s)$.

[72] Strong local dispersive estimate

[73] The functions ϕ,ψ are again generally time dependent in our applications, with derivatives decaying at least like $\dot{\nu}$. This more general case can be handled just as in the ensuing proof.

(AA): this is the expression

$$\int_T^\infty t\int_t^\infty(\nu-1)(s)\langle e^{is\mathcal{H}}\left(\begin{array}{c}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{array}\right)_{dis}(0,.),\phi\rangle\langle e^{is\mathcal{H}}\left(\begin{array}{c}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{array}\right)_{dis}(0,.),(\mathcal{H}^*)^{-k}\psi\rangle ds$$

This is straightforward to control upon invoking the spatial localization on the initial data and Theorem 2.1: we get

$$\begin{aligned}&|\int_T^\infty t\int_t^\infty(\nu-1)(s)\langle e^{is\mathcal{H}}\left(\begin{array}{c}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{array}\right)_{dis}(0,.),\phi\rangle\langle e^{is\mathcal{H}}\left(\begin{array}{c}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{array}\right)_{dis}(0,.),(\mathcal{H}^*)^{-k}\psi\rangle ds|\\ &\lesssim\int_T^\infty t\int_t^\infty s^{-\frac12+\delta_1}s^{-3}dsdt\lesssim T^{-\frac12+\delta_1},\end{aligned}$$

for δ_1 small enough, where we use the estimate derived after (5.20), with a trivial modification.

(BB): we employ schematic notation here; scalar quantities really represent vectorial quantities: this is the expression

$$\begin{aligned}&\int_T^\infty t\int_t^\infty(\nu-1)(s)\int_0^s\langle(1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)})U^{(s)}(\lambda,.)\phi,e^{-i(s-\lambda)\mathcal{H}^*}\phi_{dis}\rangle d\lambda\\ &\int_0^s\langle(1-e^{2i(\Psi-\Psi_\infty)_1(\lambda')-2i(\Psi-\Psi_\infty)_1(s)})U^{(s)}(\lambda',.)\phi,e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle d\lambda'\end{aligned}$$

For either of the integrands on the right one obtains the bound $\lambda^{-\frac32+\delta_3}(s-\lambda)^{-\frac32}$, $\lambda'^{-\frac32+\delta_3}(s-\lambda')^{-\frac32}$. This clearly suffices as long as $\lambda,\lambda'<\frac{s}{2}$. If for example $\lambda>\frac{s}{2}$, we can close provided $s-\lambda>s^{10\delta_3}$. In the opposite case, use that

$$|1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)}|<s^{-\frac12+\delta_1}s^{10\delta_3},$$

which again allows us to close. The case (AB) is handled similarly.

(BC): This is the expression

$$\begin{aligned}&\int_T^\infty t\int_t^\infty(\nu-1)(s)\int_0^s\langle(1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)})U^{(s)}(\lambda,.)\phi,e^{-i(s-\lambda)\mathcal{H}^*}\phi_{dis}\rangle d\lambda\\ &\int_0^s\langle\left(\begin{array}{c}|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda',.)\\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}(\lambda',.)}\end{array}\right)_{dis},e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle d\lambda'\end{aligned}\tag{5.30}$$

This case is much more difficult. Our method here shall make heavy use of microlocalization. The idea is to first reduce ϕ_{dis} in the local integrand (involving λ) to small $\mathcal{H}$-frequency[74]. Then either ψ_{dis} is at small frequency, too, which case is handled by exploiting an extra slack in the proof of the strong dispersive estimate, or else there is a gap between the frequency supports of these functions which forces sufficient oscillation in the s variable to render the full expression manageable. First, we observe that we may reduce to $\lambda<\frac{s}{2}$. This follows from the preceding calculation, since we gain a small power of s in the case $\lambda>\frac{s}{2}$. Now write

$$\begin{aligned}\chi_{>0}(x)e^{-i(s-\lambda)\mathcal{H}^*}\phi_{dis}&=\sum_\pm\chi_{>0}(x)\int_0^\infty e^{\pm i(s-\lambda)(\xi^2+1)}(s(\xi)e^{ix\xi}\underline{e}_\pm+\phi(x,\xi))\tilde{\mathcal{F}}_\pm(\phi_{dis})(\xi)d\xi\\ &+\sum_\pm\chi_{>0}(x)\int_{-\infty}^0 e^{\pm i(s-\lambda)(\xi^2+1)}[(e^{ix\xi}-e^{-ix\xi})\underline{e}_\pm+(1+r(-\xi))\underline{e}_\pm+\phi(x,\xi)]\tilde{\mathcal{F}}_\pm(\phi_{dis})(\xi)d\xi\end{aligned}\tag{5.31}$$

We put $\underline{e}_+:=\underline{e}=\left(\begin{array}{c}1\\0\end{array}\right)$, $\underline{e}_-:=\sigma_1\underline{e}$. Recalling that $\lambda<\frac{s}{2}$, we claim that we may build in a smooth multiplier $\phi_{<s^{-\epsilon}}(\xi)$ localizing to a dilate of the indicated region $|\xi|<s^{-\epsilon}$ in either integrand, provided $\epsilon>0$ is small enough. Indeed, if on the flip side we build in a multiplier of the form $\phi_{\geq s^{-\epsilon}}(\xi)$, integration by parts in ξ results in arbitrary gains in s, at the cost of powers of x. These, however, are absorbed by the local factor

[74] In the sense that the frequency is close to either 1 or -1.

$U\phi$ above. Thus we shall now replace $\chi_{>0}(x)e^{-i(s-\lambda)\mathcal{H}^*}\phi_{dis}$ by the sum of the above two terms with an extra cutoff $\phi_{<s^{-\epsilon}}(\xi)$ included. Denote this by $\chi_{>0}(x)e^{-i(s-\lambda)\mathcal{H}^*}\tilde{\phi}_{dis}$. Now we consider the non-local integrand. As above write

$$\chi_{>0}(x)e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis} = \sum_{\pm}\chi_{>0}(x)\int_0^\infty e^{\pm i(s-\lambda')(\xi^2+1)}(s(\xi)e^{ix\xi}\underline{e}_{\pm} + \phi(x,\xi))\frac{\tilde{\mathcal{F}}_{\pm}(\psi_{dis})(\xi)}{(\xi^2+1)^k}d\xi$$

$$+\sum_{\pm}\chi_{>0}(x)\int_{-\infty}^0 e^{\pm i(s-\lambda')(\xi^2+1)}[(e^{ix\xi}-e^{-ix\xi})\underline{e}_{\pm} + (1+r(-\xi))\underline{e}_{\pm} + \phi(x,\xi)]\frac{\tilde{\mathcal{F}}_{\pm}(\psi_{dis})(\xi)}{(\xi^2+1)^k}d\xi$$

We then distinguish between the cases $\lambda' < \frac{s}{2}$, $\lambda' > \frac{s}{2}$.

($\lambda' > \frac{s}{2}$). This case is simpler on account of the fact that no extra integration by parts is required to produce the gain of $(s-\lambda')^{-\frac{3}{2}}$, see the proof of strong local dispersive estimate. The first step consists in reducing λ' to the range $[\frac{s}{2}, s-s^{\frac{1}{10}}]$. This follows from the following simple calculation:

$$\begin{aligned}&\Big|\int_{s-s^{\frac{1}{10}}}^{s}\langle\begin{pmatrix}|\widetilde{U^{(s)}}|^4\widetilde{U^{(s)}}(\lambda',.)\\ -|\widetilde{U^{(s)}}|^4\widetilde{U^{(s)}}(\lambda',.)\end{pmatrix}, e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle d\lambda'\Big|\\
&\lesssim\int_{s-s^{\frac{1}{10}}}^{s}||\widetilde{U^{(s)}}(\lambda',.)||_{L_x^\infty}^4||\widetilde{U^{(s)}}(\lambda',.)||_{L_x^2}||e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}||_{L_x^2}d\lambda' \lesssim s^{\frac{1}{10}}s^{-2+\epsilon(\delta_2)},\end{aligned}$$

whence we obtain

$$\begin{aligned}&\Big|\int_T^\infty t\int_t^\infty(\nu-1)(s)\int_0^s\langle(1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)})U^{(s)}(\lambda,.)\phi, e^{-i(s-\lambda)\mathcal{H}^*}\tilde{\phi}_{dis}\rangle d\lambda\\
&\int_{s-s^{\frac{1}{10}}}^{s}\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda',.)\\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}(\lambda',.)}\end{pmatrix}_{dis}, e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle d\lambda'\Big|\\
&\lesssim\int_T^\infty t\int_t^\infty s^{-\frac{1}{2}+\delta_1}s^{-\frac{3}{2}}s^{\frac{1}{10}}s^{-2+\epsilon(\delta_2)}dsdt \lesssim T^{-\frac{1}{2}+\delta_1}\end{aligned}$$

Thus we now reduce to estimating

$$\begin{aligned}&\int_T^\infty t\int_t^\infty(\nu-1)(s)\int_0^{\frac{s}{2}}\langle(1-e^{i(\Psi-\Psi_\infty)_1(\lambda)-i(\Psi-\Psi_\infty)_1(s)})U^{(s)}(\lambda,.)\phi, e^{-i(s-\lambda)\mathcal{H}^*}\tilde{\phi}_{dis}\rangle d\lambda\\
&\int_{\frac{s}{2}}^{s-s^{\frac{1}{10}}}\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda',.)\\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}(\lambda',.)}\end{pmatrix}_{dis}, e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle d\lambda',\end{aligned}$$

where it is to be kept in mind that $e^{-i(s-\lambda)\mathcal{H}^*}\tilde{\phi}_{dis}$ has modified Fourier support as described above. We now mimic the proof of the strong local dispersive estimate (SLDE) for the quintilinear expression. Recall that we re-arrange the terms as follows:

$$\begin{aligned}&\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda',.)\\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}(\lambda',.)}\end{pmatrix}_{dis}, e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle\\
&=\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}, \begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle\end{aligned}$$

The first step consists in reducing both factors in the preceding expression to their dispersive part: thus write

$$\begin{aligned}&\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\-|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix},\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle\\&=\sum_i a_i\langle\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\-|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix},\xi_{k(i),proper}\rangle\eta_{i,proper},\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle\\&+\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\-|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}_{dis},\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle\end{aligned}$$

Note that

$$\begin{aligned}&|\sum_i a_i\langle\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\-|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix},\xi_{k(i),proper}\rangle\eta_{i,proper},\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle|\\&\qquad\lesssim\lambda'^{-5.(\frac{3}{2}-\delta_3)}(s-\lambda')^{-\frac{3}{2}},\end{aligned}$$

and plugging this back into the above yields an acceptable bound. Thus we now need to estimate

$$\begin{aligned}&\int_T^\infty t\int_t^\infty(\nu-1)(s)\int_0^{\frac{s}{2}}\langle(1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)})U^{(s)}(\lambda,.)\phi,e^{-i(s-\lambda)\mathcal{H}^*}\tilde{\phi}_{dis}\rangle d\lambda\\&\int_{\frac{s}{2}}^{s-s^{\frac{1}{10}}}\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\-|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}_{dis},\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle d\lambda',\end{aligned}$$

Recall from the proof of SLDE that we reformulate

$$\begin{aligned}&\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\-|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}_{dis},\chi_{>0}(x)\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle\\&=\int_{-\infty}^\infty\mathcal{F}\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4(\lambda',.)\\-\chi_{>0}(x)|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}(\xi)\overline{\tilde{\mathcal{F}}[\begin{pmatrix}\chi_{>0}(x)\overline{\tilde{U}^{(s)}(\lambda',.)}\\\chi_{>0}(x)\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}](\xi)}d\xi\end{aligned}$$

Break the integral into the contribution over $[0,\infty)$ and $(-\infty,0]$. Consider for example the latter, the former being treated similarly. We may write $\mathcal{F}$, $\tilde{\mathcal{F}}$ purely in terms of the oscillatory part: for example, consider

$$\int_{-\infty}^0\mathcal{F}\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4(\lambda',.)\\-\chi_{>0}(x)|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}(\xi)\overline{\langle[\begin{pmatrix}\chi_{>0}(x)\overline{\tilde{U}^{(s)}(\lambda',.)}\\\chi_{>0}(x)\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}],\phi(x,\xi)\rangle}d\xi \tag{5.32}$$

Estimate this by

$$\begin{aligned}&\lesssim||\mathcal{F}\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4(\lambda',.)\\-\chi_{>0}(x)|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}||_{L^2_\xi}||\langle\phi(x,\xi),[\begin{pmatrix}\chi_{>0}(x)\overline{\tilde{U}^{(s)}(\lambda',.)}\\\chi_{>0}(x)\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}]\rangle||_{L^2_\xi}\\&\lesssim\lambda'^{-\frac{3}{2}}(s-\lambda')^{-\frac{3}{2}},\end{aligned}$$

which then leads to an acceptable contribution. One argues similarly for $\mathcal{F}(...)$, and hence[75] replaces (5.32) by

$$\begin{aligned}&\int_{-\infty}^0\langle\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4(\lambda',.)\\-\chi_{>0}(x)|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix},(e^{ix\xi}-e^{-ix\xi})\underline{e}+(1+r(-\xi))e^{-ix\xi}\underline{e}\rangle\\&\overline{\langle\begin{pmatrix}\chi_{>0}(x)\overline{\tilde{U}^{(s)}(\lambda',.)}\\\chi_{>0}(x)\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis},(e^{ix\xi}-e^{-ix\xi})\underline{e}+(1+r(-\xi))e^{-ix\xi}\underline{e}\rangle}d\xi\end{aligned}$$

We first deal with the contributions of the factors $(1+r(-\xi))$, which are straightforward: note from the proof of SLDE that we distinguish between the cases $P_{><a}[\chi_{>0}(x)|\tilde{U}^{(s)}|^2]P_{><a}[|\tilde{U}^{(s)}|^2]$, where we put $a=$

[75] As usual we only consider $\mathcal{F}_+$.

$\lambda'^{-\frac{3}{4}} \sim s^{-\frac{3}{4}}$. We then put $a = \lambda'^{-\frac{3}{4}+\epsilon}$ instead, which leads to a small extra gain in s for all cases except $P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2]$. Substituting this forces $|\xi| < s^{-\frac{3}{4}+\epsilon}$, and so if either factor $(1+r(-\xi))$ occurs we again obtain a gain in s (if ϵ is small enough). Thus we now reduce to the contribution of

$$\int_{-\infty}^{0}\langle\begin{pmatrix} \chi_{>0}(x)|\tilde{U}^{(s)}|^4(\lambda',.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}, (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle$$
$$\langle\begin{pmatrix} \chi_{>0}(x)\overline{\tilde{U}^{(s)}(\lambda',.)} \\ \chi_{>0}(x)\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}, (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle d\xi$$

Arguing as before, one may reduce here to the expression

$$\int_{-\infty}^{0}\langle\begin{pmatrix} P_{<a}[\chi_{>0}(x)|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.) \\ -P_{<a}[\chi_{>0}(x)|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.)\end{pmatrix}, (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle$$
$$(5.33)\qquad \overline{\langle\begin{pmatrix} \chi_{>0}(x)\overline{\tilde{U}^{(s)}(\lambda',.)} \\ \chi_{>0}(x)\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}, (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle} d\xi,$$

where $a = \lambda'^{-\frac{3}{4}+\epsilon} \sim s^{-\frac{3}{4}+\epsilon}$. Still following the proof of SLDE, we expand

$$\chi_{>0}(x)e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis} = \chi_{>0}(x)\sum_{\pm}\int_0^{\infty} e^{\pm i(s-\lambda')(\xi^2+1)}(s(\xi)e^{ix\xi}\underline{e}_{\pm}+\phi_{\pm}(x,\xi))\frac{\mathcal{F}_{\pm}(\psi_{dis})(\xi)}{(\xi^2+1)^k}d\xi$$
$$+\chi_{>0}(x)\sum_{\pm}\int_{-\infty}^{0} e^{\pm i(s-\lambda')(\xi^2+1)}([e^{ix\xi}-e^{-ix\xi}]\underline{e}_{\pm}+(1+r(-\xi))e^{-ix\xi}\underline{e}_{\pm}+\phi_{\pm}(x,\xi))\frac{\mathcal{F}_{\pm}(\psi_{dis})(\xi)}{(\xi^2+1)^k}d\xi$$

We shall localize ξ here away from 0. For example consider the following expression:

$$\int_{-\infty}^{0} e^{\pm i(s-\lambda')(\xi^2+1)}[e^{ix\xi}-e^{-ix\xi}]\underline{e}_{\pm}\frac{\mathcal{F}_{\pm}(\psi_{dis})(\xi)}{(\xi^2+1)^k}d\xi = \frac{e^{\pm i(s-\lambda')}}{\sqrt{s-\lambda'}}\int_{-\infty}^{\infty} e^{\mp\frac{(x-y)^2}{i(s-\lambda')}}g(y)dy$$

where $g(y) = \int_{-\infty}^{0}[e^{iy\xi}-e^{-iy\xi}]\underline{e}_{\pm}\frac{\mathcal{F}_{\pm}(\psi_{dis})(\xi)}{(\xi^2+1)^k}d\xi$. It is straightforward to see that we may include a smooth multiplier $\phi_{<s^{1000}}(\xi)$ here. With this modification we then have $|g(y)| \lesssim \langle y\rangle^{-2}\log s$, whence[76] we may replace $g(y)$ by

$$\phi_{<s^{\frac{1}{10}}}(y)\int_{-\infty}^{0}\phi_{<s^{1000}}(\xi)[e^{iy\xi}-e^{-iy\xi}]\underline{e}_{\pm}\frac{\mathcal{F}_{\pm}(\psi_{dis})(\xi)}{(\xi^2+1)^k}d\xi$$

We now claim that we may further localize ξ away from 0. For this include a sharp cutoff $\chi_{<s^{-\epsilon_1}}(\xi)$, i. e. consider the contribution of

$$\tilde{g}(y) := \phi_{<s^{\frac{1}{10}}}(y)\int_{-s^{-\epsilon_1}}^{0}\phi_{<s^{1000}}(\xi)[e^{iy\xi}-e^{-iy\xi}]\underline{e}_{\pm}\frac{\mathcal{F}_{\pm}(\psi_{dis})(\xi)}{(\xi^2+1)^k}d\xi$$

We integrate by parts here, and replace this expression by

$$\phi_{<s^{\frac{1}{10}}}(y)\frac{e^{iys^{-\epsilon_1}}+e^{-iys^{-\epsilon_1}}}{iy}\underline{e}_{\pm}\frac{\mathcal{F}_{\pm}(\psi_{dis})(-s^{-\epsilon_1})}{(s^{-2\epsilon_1}+1)^k}-\phi_{<s^{\frac{1}{10}}}(y)\int_{-s^{-\epsilon_1}}^{0}\frac{[e^{iy\xi}+e^{-iy\xi}]}{iy}\underline{e}_{\pm}\partial_{\xi}[\phi_{<s^{1000}}(\xi)\frac{\mathcal{F}_{\pm}(\psi_{dis})(\xi)}{(\xi^2+1)^k}]d\xi$$

One now easily verifies that $||\tilde{g}(y)||_{L_y^1} \lesssim \log s s^{-\epsilon_1}$, which extra gain suffices to close all the other estimates upon reiterating the proof of SLDE. We now see that we may replace $g(y)$ by the expression

$$h(y) := \phi_{<s^{\frac{1}{10}}}(y)\int_{-\infty}^{-s^{-\epsilon_1}}\phi_{<s^{1000}}(\xi)[e^{iy\xi}-e^{-iy\xi}]\underline{e}_{\pm}\frac{\mathcal{F}_{\pm}(\psi_{dis})(\xi)}{(\xi^2+1)^k}d\xi$$

The Heisenberg uncertainty principle implies that this function has frequency $\gtrsim s^{-\epsilon_1}$ (for $\epsilon_1 << \frac{1}{10}$) up to errors of size s^{-N}, and hence negligible, and the same comment applies to the function

$$\frac{e^{\pm i(s-\lambda')}}{\sqrt{s-\lambda'}}\int_{-\infty}^{\infty} e^{\mp\frac{(x-y)^2}{i(s-\lambda')}}h(y)dy = \int_{-\infty}^{\infty} e^{\pm i(s-\lambda')(\xi'^2+1)}e^{ix\xi'}\hat{h}(\xi')d\xi'$$

[76] Following the proof of SLDE.

Thus we may replace[77] this function by the following, where $\phi_{\geq s^{-\epsilon_1}}$ is a smooth cutoff localizing to $|\xi'| \gtrsim s^{-\epsilon_1}$:

$$\int_{-\infty}^{\infty} e^{\pm i(s-\lambda')(\xi'^2+1)} e^{ix\xi'} \phi_{\geq s^{-\epsilon_1}}(\xi')\hat{h}(\xi')d\xi'$$

Recalling (5.33) we now need to estimate the contribution of

$$\int_{-\infty}^{0} \frac{\langle \begin{pmatrix} P_{<a}[\chi_{>0}(x)|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.) \\ -P_{<a}[\chi_{>0}(x)|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.) \end{pmatrix}, (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle}{\langle \begin{pmatrix} \chi_{>0}(x)\overline{\tilde{U}^{(s)}}(\lambda',.) \\ \chi_{>0}(x)\tilde{U}^{(s)}(\lambda',.) \end{pmatrix} \times \int_{-\infty}^{\infty} e^{\pm i(s-\lambda')(\xi'^2+1)} e^{ix\xi'} \phi_{\geq s^{-\epsilon_1}}(\xi')\hat{h}(\xi')d\xi', (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle} d\xi,$$

This then gets substituted for

$$\langle \begin{pmatrix} |\tilde{U}^{(s)}|^4(\lambda',.) \\ -|\tilde{U}^{(s)}|^4(\lambda',.) \end{pmatrix}_{dis}, \begin{pmatrix} \overline{\tilde{U}^{(s)}(\lambda',.)} \\ \tilde{U}^{(s)}(\lambda',.) \end{pmatrix} \times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle$$

in (5.30). It is now important to recall that in (5.30) we already reduced $e^{-i(s-\lambda)\mathcal{H}^*}\tilde{\phi}_{dis}$ to frequency $< s^{-\epsilon}$, see the paragraph before (5.30). Thus writing $e^{-i(s-\lambda)\mathcal{H}^*}\tilde{\phi}_{dis}$ in terms of its Fourier expansion with variable ξ and re-arranging exponentials we get the phase $e^{is(\xi^2-\xi'^2)}$ in the case of destructive resonance (and $e^{is(\xi^2+\xi'^2+2)}$for constructive resonance). If we arrange $\epsilon_1 >> \epsilon$, as we may, we have $|\xi^2 - \xi'^2| \gtrsim s^{-\epsilon_1}$. Then switch orders of integration in (5.30), first performing an the integration with respect to s. This costs s^{ϵ_1} but either demolishes the integral in λ' or else produces at least extra factors $\partial_s[\lambda_\infty[\mu - \mu_\infty]](s)$, $\frac{d}{ds}(\Psi - \Psi_\infty)_1 \sim (\nu - 1)(s)$. In order to decouple the variables ξ, ξ', notice that for $|\xi'| \lesssim s^\epsilon$ we have[78]

$$\phi_{<s^{-\epsilon}}(\xi)\phi_{\geq s^{-\epsilon_1}}(\xi')\frac{e^{i(\xi^2-\xi'^2)s}}{\xi^2 - \xi'^2} = \sum_{n,m \in s^\epsilon \mathbf{Z}^2} e^{i(\xi^2-\xi'^2)s} a_{n,m} e^{in\xi + im\xi'},$$

where one has $\sum_{n,m}[|n| + |m|]^C |a_{n,m}| \lesssim s^{C\epsilon_1}$. If one substitutes this back into (5.30) and proceeds as in the proof of SLDE, one gets an extra gain in s upon choosing ϵ_1 small enough, as desired. In detail, consider

$$\int_T^\infty t \int_t^\infty (\nu - 1)(s) \int_0^{\frac{s}{2}} \langle (1 - e^{2i(\Psi-\Psi_\infty)_1(\lambda) - 2i(\Psi-\Psi_\infty)_1(s)}) U^{(s)}(\lambda,.)\phi, e^{-i(s-\lambda)\mathcal{H}^*}\tilde{\phi}_{dis}\rangle d\lambda$$
$$\times \int_{\frac{s}{2}}^{s - s^{\frac{1}{10}}} \int_{-\infty}^{0} \frac{\langle \begin{pmatrix} P_{<a}[\chi_{>0}(x)|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.) \\ -P_{<a}[\chi_{>0}(x)|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.) \end{pmatrix}, (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle}{\times \langle \begin{pmatrix} \chi_{>0}(x)\overline{\tilde{U}^{(s)}}(\lambda',.) \\ \chi_{>0}(x)\tilde{U}^{(s)}(\lambda',.) \end{pmatrix} \times \int_{-\infty}^{\infty} e^{\pm i(s-\lambda')(\xi'^2+1)} e^{ix\xi'} \phi_{\geq s^{-\epsilon_1}}(\xi')\hat{h}(\xi')d\xi', (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle} d\xi d\lambda',$$

Then substitute (5.31) with a suitable frequency cutoff as discussed there, which amongst similar terms results in

$$\int_T^\infty t \int_t^\infty (\nu - 1)(s) \int_0^{\frac{s}{2}} \langle (1 - e^{2i(\Psi-\Psi_\infty)_1(\lambda) - 2i(\Psi-\Psi_\infty)_1(s)}) U^{(s)}(\lambda,.)\phi,$$
$$\int_{-\infty}^{0} e^{\pm i(s-\lambda)(\eta^2+1)} \phi_{<s^{-\epsilon}}(\eta)(e^{ix\eta} - e^{-ix\eta})\mathcal{F}(\phi_{dis})\eta\rangle d\lambda$$
$$\times \int_{\frac{s}{2}}^{s - s^{\frac{1}{10}}} \int_{-\infty}^{0} \frac{\langle \begin{pmatrix} P_{<a}[\chi_{>0}(x)|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.) \\ -P_{<a}[\chi_{>0}(x)|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.) \end{pmatrix}, (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle}{\times \langle \begin{pmatrix} \chi_{>0}(x)\overline{\tilde{U}^{(s)}}(\lambda',.) \\ \chi_{>0}(x)\tilde{U}^{(s)}(\lambda',.) \end{pmatrix} \times \int_{-\infty}^{\infty} e^{\pm i(s-\lambda')(\xi'^2+1)} e^{ix\xi'} \phi_{\geq s^{-\epsilon_1}}(\xi')\hat{h}(\xi')d\xi', (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle} d\xi d\lambda',$$

---

[77] Generating negligible error terms

[78] We may easily reduce to $|\xi'| \lesssim s^\epsilon$, see e.g. the argument for Lemma 5.7

Rewrite this as

$$\int_T^\infty t\int_t^\infty (\nu-1)(s)\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^\infty\int_0^\infty\int_0^\infty e^{is(\pm\eta^2\pm\xi'^2)}\Psi(s,\lambda,\lambda',\eta,\xi,\xi')d\lambda d\lambda' d\xi' d\eta d\xi ds dt$$

Now perform an integration by parts in s, after switching the orders of integration, then restore the original order of integration:

$$\begin{aligned}&\int_T^\infty t\int_t^\infty (\nu-1)(s)\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^\infty\int_0^\infty\int_0^\infty e^{is(\pm\eta^2\pm\xi'^2)}\Psi(s,\lambda,\lambda',\eta,\xi,\xi')d\eta d\xi d\xi' d\lambda d\lambda' ds dt\\
&=-\int_T^\infty t(\nu-1)(t)\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^\infty\int_0^\infty\int_0^\infty \frac{e^{it(\pm\eta^2\pm\xi'^2)}}{i(\pm\eta^2\pm\xi'^2)}\Psi(t,\lambda,\lambda',\eta,\xi,\xi')d\eta d\xi d\xi' d\lambda d\lambda' ds dt\\
&+\int_T^\infty t\int_t^\infty \dot{\nu}(s)\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^\infty\int_0^\infty\int_0^\infty \frac{e^{is(\pm\eta^2\pm\xi'^2)}}{i(\pm\eta^2\pm\xi'^2)}\Psi(s,\lambda,\lambda',\eta,\xi,\xi')d\eta d\xi d\xi' d\lambda d\lambda' ds dt\\
&+\int_T^\infty t\int_t^\infty (\nu-1)(s)\int_{-\infty}^0\int_{-\infty}^0\int_{-\infty}^\infty\int_0^\infty\int_0^\infty \frac{e^{is(\pm\eta^2\pm\xi'^2)}}{i(\pm\eta^2\pm\xi'^2)}\partial_s\Psi(s,\lambda,\lambda',\eta,\xi,\xi')d\eta d\xi d\xi' d\lambda d\lambda' ds dt\end{aligned}$$

We have $|\xi'^2\pm\eta^2|\gtrsim s^{-\epsilon_1}$ on the support of each integrand. If one then decouples the variables ξ', η as outlined above and then proceeds as in the proof of SLDE, one checks that each of these terms can be bounded by $\lesssim T^{-\frac12+\delta_1-\mu(\epsilon_1)}$ upon choosing ϵ_1 small enough, which suffices. This concludes the case $\lambda'>\frac{s}{2}$.

($\lambda'<\frac{s}{2}$), still in case (BC). This is the expression

$$\begin{aligned}&\int_T^\infty t\int_t^\infty(\nu-1)(s)\int_0^{\frac{s}{2}}\langle(1-e^{2i(\Psi-\Psi_\infty)_1(\lambda)-2i(\Psi-\Psi_\infty)_1(s)})U^{(s)}(\lambda,.)\phi, e^{-i(s-\lambda)\mathcal{H}^*}\tilde{\phi}_{dis}\rangle d\lambda\\
&\int_0^{\frac{s}{2}}\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix},\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle d\lambda',\end{aligned}\tag{5.34}$$

We start again by reducing

$$\langle\chi_{>0}\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix},\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle,\tag{5.35}$$

to

$$\langle\chi_{>0}\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -|\tilde{U}^{(s)}|^4(\lambda,.)\end{pmatrix}_{dis},\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle,\tag{5.36}$$

This follows from

$$\sum_i|\langle\chi_{>0}\langle\begin{pmatrix}|\tilde{U}^{(s)}|^4\\ -|\tilde{U}^{(s)}|^4\end{pmatrix},\xi_i\rangle\eta_i,\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle|\lesssim\lambda'^{-5(\frac32-\delta_3)}(s-\lambda')^{-\frac32},\tag{5.37}$$

Now replace (5.36) by

$$\int_{-\infty}^\infty\mathcal{F}[\chi_{>0}\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -|\tilde{U}^{(s)}|^4(\lambda,.)\end{pmatrix}](\xi)\overline{\tilde{\mathcal{F}}[\chi_{>0}(x)\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}\psi_{dis}](\xi)}d\xi\tag{5.38}$$

We shall again simplify the Fourier transform here: for example, consider the contribution of

$$\int_{-\infty}^\infty\langle\chi_{>0}\begin{pmatrix}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -|\tilde{U}^{(s)}|^4(\lambda,.)\end{pmatrix},\phi(x,\xi)\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}(x)\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}](\xi)}d\xi\tag{5.39}$$

Proceeding as in the proof of SLDE one bounds this by $\lesssim \lambda'^{-4(\frac{3}{2}-\delta_3)}\lambda'(s-\lambda')^{-\frac{3}{2}}$, which is more than enough. Further, for example the contribution of

(5.40)
$$\int_{-\infty}^{0}\langle\chi_{>0}\left(\begin{array}{c}|\tilde{U}^{(s)}|^4(\lambda',.)\\-|\tilde{U}^{(s)}|^4(\lambda,.)\end{array}\right),(1+r(-\xi))e^{-ix\xi}\underline{e}\rangle\overline{\tilde{\mathcal{F}}[\chi_{>0}(x)\left(\begin{array}{c}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{array}\right)\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}](\xi)}d\xi$$

is treated like in the case $\lambda'>\frac{s}{2}$ (one doesn't gain in s but in λ'). Thus focusing on the more difficult reflection part, we need to estimate the contribution of

$$\begin{aligned}&\int_{-\infty}^{0}\langle\chi_{>0}\left(\begin{array}{c}|\tilde{U}^{(s)}|^4(\lambda',.)\\-|\tilde{U}^{(s)}|^4(\lambda,.)\end{array}\right),(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle\\&\times\overline{\langle[\chi_{>0}(x)\left(\begin{array}{c}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{array}\right)\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}],(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle}d\xi\end{aligned}\tag{5.41}$$

Arguing as in the case $\lambda'>\frac{s}{2}$, we may reduce this expression further to

$$\begin{aligned}&\int_{-\infty}^{0}\langle\left(\begin{array}{c}P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.)\\-P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.)\end{array}\right),(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle\\&\times\overline{\langle[\chi_{>0}(x)\left(\begin{array}{c}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{array}\right)\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}],(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle}d\xi\end{aligned}$$

where $a=\lambda'^{-\frac{3}{4}+\epsilon}$. We shall next show that we may localize the Fourier support of $e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}$ away from zero, in which case we can conclude as in the case $\lambda'>\frac{s}{2}$, exploiting the frequency separation in order to perform an integration by parts in s. Recall from the proof of the SLDE that in this case, we need to perform an extra integration by parts in the frequency variable in order to obtain the gain $(s-\lambda')^{-\frac{3}{2}}$. More precisely, in the expression

$$\langle\chi_{>0}\left(\begin{array}{c}|\tilde{U}^{(s)}|^4\\-|\tilde{U}^{(s)}|^4\end{array}\right)_{dis},\left(\begin{array}{c}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{array}\right)\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle,$$

we replace the expression

$$\chi_{>0}(x)\left(\begin{array}{c}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{array}\right)\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}$$

by

$$\begin{aligned}&\chi_{>0}(x)\left(\begin{array}{c}x\overline{\tilde{U}^{(s)}(\lambda',.)}\\x\tilde{U}^{(s)}(\lambda',.)\end{array}\right)\times\frac{1}{s-\lambda'}\int_{-\infty}^{0}e^{-i(s-\lambda')(\xi^2+1)}(e^{ix\xi}+e^{-ix\xi})\underline{e}[\frac{\mathcal{F}(\psi_{dis})(\xi)}{(\xi^2+1)^k\xi}]d\xi\\&\chi_{>0}(x)\left(\begin{array}{c}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{array}\right)\times\frac{1}{s-\lambda'}\int_{-\infty}^{0}e^{-i(s-\lambda')(\xi^2+1)}(e^{ix\xi}-e^{-ix\xi})\underline{e}\partial_\xi[\frac{\mathcal{F}(\psi_{dis})(\xi)}{(\xi^2+1)^k\xi}]d\xi\\&\chi_{>0}(x)\left(\begin{array}{c}x\overline{\tilde{U}^{(s)}(\lambda',.)}\\x\tilde{U}^{(s)}(\lambda',.)\end{array}\right)\times\frac{1}{s-\lambda'}\int_{0}^{\infty}e^{-i(s-\lambda')(\xi^2+1)}s(\xi)e^{ix\xi}\underline{e}\frac{\mathcal{F}\psi_{dis}(\xi)}{(\xi^2+1)^k\xi}d\xi\\&\chi_{>0}(x)\left(\begin{array}{c}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{array}\right)\times\frac{1}{s-\lambda'}\int_{0}^{\infty}e^{-i(s-\lambda')(\xi^2+1)}\partial_\xi s(\xi)e^{ix\xi}\underline{e}\frac{\mathcal{F}\psi_{dis}(\xi)}{(\xi^2+1)^k\xi}d\xi\end{aligned}\tag{5.42}$$

as well as the expressions

$$\chi_{>0}(x)\left(\begin{array}{c}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{array}\right)\times\frac{1}{s-\lambda'}\int_{-\infty}^{0}e^{-i(s-\lambda')(\xi^2+1)}e^{-ix\xi}\partial_\xi[1+r(-\xi)]\underline{e}[\frac{\mathcal{F}(\psi_{dis})(\xi)}{(\xi^2+1)^k\xi}]d\xi$$

$$\chi_{>0}(x)\left(\begin{array}{c}\overline{\tilde{U}^{(s)}(\lambda',.)}\\\tilde{U}^{(s)}(\lambda',.)\end{array}\right)\times\frac{1}{s-\lambda'}\int_{-\infty}^{0}e^{-i(s-\lambda')(\xi^2+1)}\partial_\xi[\phi(x,\xi)[\frac{\mathcal{F}(\psi_{dis})(\xi)}{(\xi^2+1)^k\xi}]]d\xi$$

as well as similar terms which can be treated identically. The last term but one here is equivalent to the last term but two for all intents and purposes. Moreover, the last term can be treated by the same argument as for the last term but two, so we shall now consider the four terms after and including (5.42). Start with (5.42): we have

$$\int_{-\infty}^{0} e^{-i(s-\lambda')(\xi^2+1)}(e^{ix\xi}+e^{-ix\xi})\underline{e}[\frac{\mathcal{F}(\psi_{dis})(\xi)}{(\xi^2+1)^k\xi}]d\xi = \frac{1}{\sqrt{s-\lambda'}}\int_{-\infty}^{\infty} e^{\frac{(x-y)^2}{i(s-\lambda')}}g(y)dy,$$

where $g(y) = \int_{-\infty}^{0}(e^{iy\xi}+e^{-iy\xi})\underline{e}[\frac{\mathcal{F}(\phi_{dis})(\xi)}{(\xi^2+1)^k\xi}]d\xi$ satisfies $|g(y)| \lesssim \langle y\rangle^{-2}$, whence we may replace it by $\tilde{g}(y) = \phi_{<\lambda'^{\frac{1}{10}}}(y)g(y)$. Now we specialize this further and consider the contribution of

$$h(y) := \phi_{<\lambda'^{\frac{1}{10}}}(y)\int_{-\infty}^{0}\phi_{<\lambda'^{-\epsilon_1}}(\xi)(e^{iy\xi}+e^{-iy\xi})\underline{e}[\frac{\mathcal{F}(\phi_{dis})(\xi)}{(\xi^2+1)^k\xi}]d\xi$$

By the Heisenberg principle, it has frequency in the interval $[0, \lambda'^{-\frac{\epsilon_1}{2}}]$ up to errors of size λ'^{-N}, which we may neglect. Now consider the bracket

$$\langle (e^{ix\xi}-e^{-ix\xi})\underline{e}, [\chi_{>0}(x)\begin{pmatrix} x\overline{\tilde{U}^{(s)}(\lambda',.)} \\ x\tilde{U}^{(s)}(\lambda',.)\end{pmatrix} \times \frac{1}{(\sqrt{s-\lambda'})^3}\int_{-\infty}^{\infty} e^{\frac{(x-y)^2}{i(s-\lambda')}}h(y)dy]\rangle,$$

where we have the restriction $|\xi| \lesssim \lambda'^{-\frac{3}{4}+\epsilon}$. We can then rewrite this as

$$\chi_{\lesssim\lambda'^{-\frac{3}{4}+\epsilon}}(\xi)\langle (e^{ix\xi}-e^{-ix\xi})\underline{e}, [P_{<\lambda'^{-\frac{\epsilon_1}{2}}}[\begin{pmatrix} \chi_{>0}(x)x\overline{\tilde{U}^{(s)}(\lambda',.)} \\ \chi_{>0}(x)x\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}] \times \frac{1}{(\sqrt{s-\lambda'})^3}\int_{-\infty}^{\infty} e^{\frac{(x-y)^2}{i(s-\lambda')}}h(y)dy]\rangle,$$

Then we use Littlewood-Paley dichotomy in order to get

$$\begin{aligned}P_{<\lambda'^{-\frac{\epsilon_1}{2}}}[\begin{pmatrix} \chi_{>0}(x)x\overline{\tilde{U}^{(s)}(\lambda',.)} \\ \chi_{>0}(x)x\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}] &= P_{<\lambda'^{-\frac{\epsilon_1}{2}}}[\begin{pmatrix} \chi_{>0}(x)P_{<\lambda'^{-\frac{\epsilon_1}{2}}+10}[x\overline{\tilde{U}^{(s)}}](\lambda',.) \\ \chi_{>0}(x)P_{<\lambda'^{-\frac{\epsilon_1}{2}}+10}[x\tilde{U}^{(s)}](\lambda',.)\end{pmatrix}] \\ &+ P_{<\lambda'^{-\frac{\epsilon_1}{2}}}[\begin{pmatrix} P_{\geq\lambda'^{-\frac{\epsilon_1}{2}}}[\chi_{>0}(x)][P_{\geq\lambda'^{-\frac{\epsilon_1}{2}}+10}][x\overline{\tilde{U}^{(s)}}](\lambda',.) \\ P_{\geq\lambda'^{-\frac{\epsilon_1}{2}}}[\chi_{>0}(x)]P_{\geq\lambda'^{-\frac{\epsilon_1}{2}}+10}[x\tilde{U}^{(s)}](\lambda',.)\end{pmatrix}]\end{aligned}$$

We first consider the contribution from the 2nd term on the right. We substitute this back into (5.41) undo the Fourier transform using Plancherel's Theorem, and estimate this by

$$\begin{aligned}&\lesssim ||\chi_{>0}\begin{pmatrix} P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.) \\ -P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2](\lambda',.)P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.)\end{pmatrix}||_{L_x^\infty}||P_{\geq\lambda'^{-\frac{\epsilon_1}{2}}}[\chi_{>0}(x)]||_{L_x^1} \\ &||x\tilde{U}^{(s)}(\lambda',.)||_{L_x^\infty}||\frac{1}{(\sqrt{s-\lambda'})^3}\int_{-\infty}^{\infty} e^{\frac{(x-y)^2}{i(s-\lambda')}}h(y)dy||_{L_x^\infty} \lesssim \lambda'^{-\frac{3}{2}+\epsilon_1}(s-\lambda')^{-\frac{3}{2}},\end{aligned}$$

which is then seen to lead to an acceptable contribution upon substitution into (5.34). Thus we may now focus on the contribution of

$$\langle (e^{ix\xi}-e^{-ix\xi})\underline{e}, [\chi_{>0}(x)\begin{pmatrix} P_{<\lambda'^{-\frac{\epsilon_1}{2}}+10}[x\overline{\tilde{U}^{(s)}}](\lambda',.) \\ P_{<\lambda'^{-\frac{\epsilon_1}{2}}+10}[x\tilde{U}^{(s)}](\lambda',.)\end{pmatrix} \times \frac{1}{(\sqrt{s-\lambda'})^3}\int_{-\infty}^{\infty} e^{\frac{(x-y)^2}{i(s-\lambda')}}h(y)dy]\rangle,$$

always keeping in mind that $|\xi| < \lambda'^{-\frac{3}{4}+\epsilon}$. We now replicate the proof of SLDE for the low-low case (keep in mind that the full expression we estimate is (5.41)). Thus we write ($p = -i\partial_x$)

$$P_{<\lambda'^{-\frac{\epsilon_1}{2}}+10}[x\tilde{U}^{(s)}](t,.) = P_{<\lambda'^{-\frac{\epsilon_1}{2}}+10}[(x+2ipt)\tilde{U}^{(s)}](t,.) - P_{<\lambda'^{-\frac{\epsilon_1}{2}}+10}[2ipt\tilde{U}^{(s)}](t,.)$$

Then we have

$$[i\partial_t + \triangle]P_{<\lambda'^{-\frac{\epsilon_1}{2}}+10}\nabla U(t,.) = P_{<\lambda'^{-\frac{\epsilon_1}{2}}+10}\nabla[VU + \ldots + |U|^4U]$$

just as in the proof of the strong local dispersive estimate. Now one further manipulates the expressions on the right just as in the proof of SLDE. Note that the operator $P_{<\lambda'^{-\frac{\epsilon_1}{2}}+10}\nabla$ will smear out the supports a bit, but this is easily seen to be harmless. Of course one gains $\lambda'^{-\frac{\epsilon_1}{2}}$ in the process, which overcomes any small losses in the proof of SLDE. One can now restrict to $|\xi| > \lambda'^{-\epsilon_1}$, i. e. include a multiplier $\phi_{\geq \lambda'^{-\epsilon_1}}$ in the definition of $h(y)$, and then finish the argument just as in the case $\lambda' > \frac{s}{2}$. This concludes estimating the contribution from the term (5.42). The contribution of the third term in that list is treated analogously. We now turn to the contribution of the 2nd term there, i. e. the expression

$$\langle \chi_{>0} \begin{pmatrix} |\tilde{U}^{(s)}|^4 \\ -|\tilde{U}^{(s)}|^4 \end{pmatrix}_{dis} (\lambda', .),$$

$$\chi_{>0}(x) \begin{pmatrix} \overline{\tilde{U}^{(s)}}(\lambda', .) \\ \tilde{U}^{(s)}(\lambda', .) \end{pmatrix} \times \frac{1}{s-\lambda'} \int_{-\infty}^{0} e^{-i(s-\lambda')(\xi^2+1)} (e^{ix\xi} - e^{-ix\xi}) \underline{e} \partial_\xi [\frac{\mathcal{F}(\psi_{dis})(\xi)}{(\xi^2+1)^k \xi}] d\xi \rangle$$

But this is easily seen to be estimable by

$$\lesssim \lambda'^{-\frac{3}{2}} (s-\lambda')^{-\frac{3}{2}},$$

which upon substitution into (5.34) yields an acceptable contribution. The fourth term after (5.42) is handled analogously. We are done with the case (BC). Clearly the case (AC) can be handled analogously.

(CC): the most difficult case. This is the expression

$$\begin{aligned} &\int_T^\infty t \int_t^\infty (\nu-1)(s) \int_0^s \langle \begin{pmatrix} |\tilde{U}^{(s)}|^4 \tilde{U}^{(s)}(\lambda, .) \\ -|\tilde{U}^{(s)}|^4 \overline{\tilde{U}^{(s)}(\lambda, .)} \end{pmatrix}_{dis}, e^{-i(s-\lambda)\mathcal{H}^*} \phi_{dis} \rangle d\lambda \\ &\qquad \int_0^s \langle \begin{pmatrix} |\tilde{U}^{(s)}|^4 \tilde{U}^{(s)}(\lambda', .) \\ -|\tilde{U}^{(s)}|^4 \overline{\tilde{U}^{(s)}(\lambda', .)} \end{pmatrix}_{dis}, e^{-i(s-\lambda')\mathcal{H}^*} (\mathcal{H}^*)^{-k} \psi_{dis} \rangle d\lambda' \end{aligned} \tag{5.43}$$

Start with the case $\max\{\lambda, \lambda'\} < \frac{s}{2}$. We may restrict integration to the range $\lambda > \lambda'$. Rearrange either of the factors in the integral as

$$\langle \chi_{>0} \begin{pmatrix} |\tilde{U}^{(s)}|^4(\lambda', .) \\ -|\tilde{U}^{(s)}|^4(\lambda', .) \end{pmatrix}, \begin{pmatrix} \overline{\tilde{U}^{(s)}(\lambda', .)} \\ \tilde{U}^{(s)}(\lambda', .) \end{pmatrix} \times e^{-i(s-\lambda')\mathcal{H}^*} (\mathcal{H}^*)^{-k} \psi_{dis} \rangle,$$

$$\langle \chi_{>0} \begin{pmatrix} |\tilde{U}^{(s)}|^4(\lambda, .) \\ -|\tilde{U}^{(s)}|^4(\lambda, .) \end{pmatrix}, \begin{pmatrix} \overline{\tilde{U}^{(s)}(\lambda, .)} \\ \tilde{U}^{(s)}(\lambda, .) \end{pmatrix} \times e^{-i(s-\lambda)\mathcal{H}^*} \phi_{dis} \rangle,$$

As usual we first need to reduce both factors in either bracket to their dispersive part. This time, though, we have to analyze each constituent more carefully, since they all interact with each other. Thus we now write

$$\chi_{>0} \begin{pmatrix} |\tilde{U}^{(s)}|^4(\lambda', .) \\ -|\tilde{U}^{(s)}|^4(\lambda', .) \end{pmatrix} = \sum_i a_i \langle \chi_{>0} \begin{pmatrix} |\tilde{U}^{(s)}|^4(\lambda', .) \\ -|\tilde{U}^{(s)}|^4(\lambda', .) \end{pmatrix}, \xi_{k(i)} \rangle \eta_i + \begin{pmatrix} |\chi_{>0} \tilde{U}^{(s)}|^4(\lambda', .) \\ -\chi_{>0} |\tilde{U}^{(s)}|^4(\lambda', .) \end{pmatrix}_{dis} := \alpha + \beta$$

First consider the contribution from $\alpha(\lambda', .)$, i. e. the expression

$$\langle \alpha(\lambda', .), \begin{pmatrix} \overline{\tilde{U}^{(s)}(\lambda', .)} \\ \tilde{U}^{(s)}(\lambda', .) \end{pmatrix} \times e^{-i(s-\lambda')\mathcal{H}^*} (\mathcal{H}^*)^{-k} \psi_{dis} \rangle$$

As usual we expand

$$e^{-i(s-\lambda')\mathcal{H}^*} (\mathcal{H}^*)^{-k} \psi_{dis} = \sum_{\pm} \int_{-\infty}^{\infty} e^{\pm i(s-\lambda')(\xi^2+1)} e_{\pm}(x, \xi) \frac{\mathcal{F}_{\pm}(\psi_{dis})}{(\xi^2+1)^k} d\xi$$

We claim that we may sneak in a smooth cutoff $\phi_{<s^{-\epsilon}}(\xi)$ into this integrand, which we then denote as $e^{-i(s-\lambda')\mathcal{H}^*} (\mathcal{H}^*)^{-k} \tilde{\psi}_{dis}$. This is because integration by parts in ξ costs in addition to s^ϵ at most $\max\{|x|, \xi\}$, and for α we may assume $|x|$ to be bounded, whence choosing ϵ small enough results in a gain in s. Of course we use $\lambda' < \frac{s}{2}$ here. The same comment applies to $\alpha(\lambda, .)$. Our strategy shall be to achieve a localization away from zero for the frequencies of $e^{-i(s-\lambda)\mathcal{H}^*} \phi_{dis}$, $e^{-i(s-\lambda)\mathcal{H}^*} (\mathcal{H}^*)^{-k} \psi_{dis}$, occuring in the contribution from $\beta(\lambda, .)$, $\beta(\lambda', .)$. This ensures that $\alpha(\lambda', .)$ and $\beta(\lambda, .)$ etc interact weakly. We now distinguish between the following cases:

$(\alpha\alpha)$: this is the expression

(5.44)

$$\int_T^\infty t\int_t^\infty (\nu-1)(s)\int_0^{\frac{s}{2}}\langle\langle\chi_{>0}\begin{pmatrix} |\tilde{U}^{(s)}|^4(\lambda,.)\\ -|\tilde{U}^{(s)}|^4(\lambda,.)\end{pmatrix},\xi_i\rangle\eta_i,\begin{pmatrix}\overline{\tilde{U}^{(s)}}(\lambda,.)\\ \tilde{U}^{(s)}(\lambda,.)\end{pmatrix}\times e^{-i(s-\lambda)\mathcal{H}^*}\tilde{\phi}_{dis}\rangle d\lambda$$
$$\int_0^{\frac{s}{2}}\langle\langle\chi_{>0}\begin{pmatrix} |\tilde{U}^{(s)}|^4(\lambda',.)\\ -|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix},\xi_i\rangle\eta_i,\begin{pmatrix}\overline{\tilde{U}^{(s)}}(\lambda',.)\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\tilde{\psi}_{dis}\rangle d\lambda'$$

We can easily estimate this by

$$\lesssim\int_T^\infty t\int_t^\infty s^{-\frac12+\delta_1}\int_0^{\frac{s}{2}}\lambda^{-5(\frac32-\delta_3)}(s-\lambda)^{-\frac32}d\lambda\int_0^{\frac{s}{2}}\lambda'^{-5(\frac32-\delta_3)}(s-\lambda')^{-\frac32}d\lambda'\lesssim T^{-\frac12+\delta_1}$$

$(\alpha\beta)$: the expression

(5.45)

$$\int_T^\infty t\int_t^\infty (\nu-1)(s)\int_0^{\frac{s}{2}}\langle\langle\chi_{>0}\begin{pmatrix} |\tilde{U}^{(s)}|^4(\lambda,.)\\ -|\tilde{U}^{(s)}|^4(\lambda,.)\end{pmatrix},\xi_i\rangle\eta_i,\begin{pmatrix}\overline{\tilde{U}^{(s)}}(\lambda,.)\\ \tilde{U}^{(s)}(\lambda,.)\end{pmatrix}\times e^{-i(s-\lambda)\mathcal{H}^*}\tilde{\phi}_{dis}\rangle d\lambda$$
$$\int_0^{\frac{s}{2}}\langle\begin{pmatrix} \chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}_{dis},\begin{pmatrix}\overline{\tilde{U}^{(s)}}(\lambda',.)\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle d\lambda'$$

To proceed, we restrict the frequency of $e^{-i(s-\lambda')\mathcal{H}^*}\psi_{dis}$ away from 0. The procedure for this is identical to the one outlined in case (BC). Having achieved frequency separation, we have of course achieved rapid oscillation in s, whence we can close this case like at the end of case (BC), by integration by parts in s. The case $(\beta\alpha)$ is handled analogously.

$(\beta\beta)$: the expression

$$\int_T^\infty t\int_t^\infty (\nu-1)(s)\int_0^{\frac{s}{2}}\langle\langle\begin{pmatrix} \chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\end{pmatrix}_{dis},\begin{pmatrix}\overline{\tilde{U}^{(s)}}(\lambda,.)\\ \tilde{U}^{(s)}(\lambda,.)\end{pmatrix}\times e^{-i(s-\lambda)\mathcal{H}^*}\phi_{dis}\rangle d\lambda$$
$$\int_0^{\frac{s}{2}}\langle\begin{pmatrix} \chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}_{dis},\begin{pmatrix}\overline{\tilde{U}^{(s)}}(\lambda',.)\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}\rangle d\lambda' \quad (5.46)$$

As before we mimic the proof of SLDE. Thus we perform an integration by parts in the Fourier representation for $e^{-i(s-\lambda)\mathcal{H}^*}\phi_{dis}$ etc and produce the following list of terms provided the integration by parts results in a loss of x. Call this list β_1:

$$\chi_{>0}(x)\begin{pmatrix} x\overline{\tilde{U}^{(s)}(\lambda',.)}\\ x\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times\frac{1}{s-\lambda'}\int_{-\infty}^0 e^{-i(s-\lambda')(\xi^2+1)}(e^{ix\xi}+e^{-ix\xi})\underline{e}[\frac{\mathcal{F}(\psi_{dis})(\xi)}{(\xi^2+1)^k\xi}]d\xi \quad (5.47)$$

$$\chi_{>0}(x)\begin{pmatrix} x\overline{\tilde{U}^{(s)}(\lambda',.)}\\ x\tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times\frac{1}{s-\lambda'}\int_0^\infty e^{-i(s-\lambda')(\xi^2+1)}s(\xi)e^{ix\xi}\underline{e}\frac{\mathcal{F}\psi_{dis}(\xi)}{(\xi^2+1)^k\xi}d\xi$$

$$\chi_{>0}(x)\begin{pmatrix} x\overline{\tilde{U}^{(s)}(\lambda,.)}\\ x\tilde{U}^{(s)}(\lambda,.)\end{pmatrix}\times\frac{1}{s-\lambda}\int_{-\infty}^0 e^{-i(s-\lambda)(\xi^2+1)}(e^{ix\xi}+e^{-ix\xi})\underline{e}[\frac{\mathcal{F}(\phi_{dis})(\xi)}{\xi}]d\xi \quad (5.48)$$

$$\chi_{>0}(x)\begin{pmatrix} x\overline{\tilde{U}^{(s)}(\lambda,.)}\\ x\tilde{U}^{(s)}(\lambda,.)\end{pmatrix}\times\frac{1}{s-\lambda}\int_0^\infty e^{-i(s-\lambda)(\xi^2+1)}s(\xi)e^{ix\xi}\underline{e}\frac{\mathcal{F}\phi_{dis}(\xi)}{\xi}d\xi$$

These get complemented by the following terms, which we refer to as β_2:

$$\chi_{>0}(x)\begin{pmatrix} \overline{\tilde{U}^{(s)}(\lambda',.)}\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times\frac{1}{s-\lambda'}\int_{-\infty}^0 e^{-i(s-\lambda')(\xi^2+1)}(e^{ix\xi}-e^{-ix\xi})\underline{e}\partial_\xi[\frac{\mathcal{F}(\psi_{dis})(\xi)}{(\xi^2+1)^k\xi}]d\xi \quad (5.49)$$

$$\chi_{>0}(x)\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times\frac{1}{s-\lambda'}\int_0^\infty e^{-i(s-\lambda')(\xi^2+1)}\partial_\xi s(\xi)e^{ix\xi}\underline{e}\frac{\mathcal{F}\psi_{dis}(\xi)}{(\xi^2+1)^k\xi}d\xi$$

$$\chi_{>0}(x)\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda,.)}\\ \tilde{U}^{(s)}(\lambda,.)\end{pmatrix}\times\frac{1}{s-\lambda}\int_{-\infty}^0 e^{-i(s-\lambda)(\xi^2+1)}(e^{ix\xi}-e^{-ix\xi})\underline{e}\partial_\xi[\frac{\mathcal{F}(\phi_{dis})(\xi)}{\xi}]d\xi$$

$$\chi_{>0}(x)\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda,.)}\\ \tilde{U}^{(s)}(\lambda,.)\end{pmatrix}\times\frac{1}{s-\lambda}\int_0^\infty e^{-i(s-\lambda)(\xi^2+1)}\partial_\xi s(\xi)e^{ix\xi}\underline{e}\frac{\mathcal{F}\phi_{dis}(\xi)}{\xi}d\xi$$

$(\beta_2\beta_2)$: this is an expression of the form

$$\int_T^\infty t\int_t^\infty(\nu-1)(s)\int_0^{\frac{s}{2}}\langle\begin{pmatrix}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}_{dis},\beta_2(\lambda',.)\rangle d\lambda'\int_0^{\frac{s}{2}}\langle\begin{pmatrix}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\end{pmatrix}_{dis},\beta_2(\lambda,.)\rangle d\lambda$$

where $\beta_2(\lambda',.),\beta_2(\lambda,.)$ stand for certain terms of the 2nd list. This type of interaction is easy to control: one bounds this by

$$\lesssim\int_T^\infty t\int_t^\infty s^{-\frac{1}{2}+\delta_1}\int_0^{\frac{s}{2}}\lambda'^{-\frac{3}{2}}(s-\lambda')^{-\frac{3}{2}}d\lambda'\int_0^\lambda\lambda^{-\frac{3}{2}}(s-\lambda)^{-\frac{3}{2}}d\lambda\lesssim T^{-\frac{1}{2}+\delta_1}$$

$(\beta_1\beta_2)$: this is an expression of the form

$$\int_T^\infty t\int_t^\infty(\nu-1)(s)\int_0^{\frac{s}{2}}\langle\begin{pmatrix}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\end{pmatrix}_{dis},\beta_1(\lambda,.)\rangle d\lambda\int_0^{\frac{s}{2}}\langle\begin{pmatrix}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}_{dis},\beta_2(\lambda',.)\rangle d\lambda'$$

Assume for example(the other cases being treated by exact analogy) that $\beta_2(\lambda',.)$ has the following form:

$$\beta_2(\lambda',.)=\chi_{>0}(x)\begin{pmatrix}\overline{\tilde{U}^{(s)}(\lambda',.)}\\ \tilde{U}^{(s)}(\lambda',.)\end{pmatrix}\times\frac{1}{s-\lambda'}\int_0^\infty e^{-i(s-\lambda')(\xi^2+1)}\partial_\xi s(\xi)e^{ix\xi}\underline{e}\frac{\mathcal{F}\phi_{dis}(\xi)}{\xi}d\xi$$

Note that on account of

$$|\langle\begin{pmatrix}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}_{dis},\beta_2(\lambda',.)\rangle|\lesssim\lambda'^{-\frac{3}{2}}(s-\lambda')^{-\frac{3}{2}},$$

we may assume that $\lambda'<s^\epsilon$ for a small $\epsilon>0$. But then on account of the pseudo-conformal almost conservation we may apply a localizer $\phi_{<s^{2\epsilon}}(x)$ to the quadrilinear term: indeed, we have

$$\begin{aligned}&|\langle\begin{pmatrix}\chi_{>0}\phi_{\geq s^{2\epsilon}}(x)|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}\phi_{\geq s^{2\epsilon}}(x)|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}_{dis},\beta_2(\lambda',.)\rangle|\\ &\lesssim||\frac{1}{|x|}\phi_{\geq s^{2\epsilon}}(|x|)[(x+2ip\lambda')U-2ip\lambda'U]||_{L_x^2}||U(\lambda,.)||_{L_x^2}||U(\lambda',.)||^3_{L_x^\infty}(s-\lambda')^{-\frac{3}{2}}\\ &\lesssim s^{-\epsilon+\delta_2}\lambda'^{-\frac{3}{2}}(s-\lambda')^{-\frac{3}{2}},\end{aligned}$$

which leads to an acceptable contribution above. Finally, we may reduce the frequency ξ in the relation defining $\beta_2(\lambda',.)$ above to absolute size $<s^{-\epsilon}$ by inclusion of a suitable smooth cutoff $\phi_{<s^{-\epsilon}}(\xi)$. This is since upon including a smooth cutoff $\phi_{\geq s^{-\epsilon}}(\xi)$ for suitably small ϵ results in an expression which can be integrated by parts in ξ, resulting in losses of at most $\max\{|x|,s^\epsilon\}s^\epsilon$ for each integration while resulting in a gain of $s-\lambda'$. Choosing ϵ small enough results in arbitrary gains in s. Next, we consider $\beta_1(\lambda,.)$. Using the same argument as in case (BC), we reduce the frequency to size $>\lambda^{-\epsilon_1}$. But then we have again achieved frequency separation and can integrate by parts in s. The case $(\beta_2\beta_1)$ is simpler, as one gains $\lambda^{-\frac{1}{2}}$ which suffices (since $\lambda>\lambda'$).

$(\beta_1\beta_1)$: This is an expression of the form

$$\int_T^\infty t\int_t^\infty(\nu-1)(s)\int_0^{\frac{s}{2}}\langle\begin{pmatrix}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix}_{dis},\beta_1(\lambda',.)\rangle d\lambda'\int_0^{\frac{s}{2}}\langle\begin{pmatrix}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\end{pmatrix}_{dis},\beta_1(\lambda,.)\rangle d\lambda$$

Keep in mind that we assume $\lambda > \lambda'$. Use the distorted Plancherel's Theorem 2.3 to rewrite this as

$$\sum_{\pm,\pm}\int_T^\infty t\int_t^\infty(\nu-1)(s)\int_0^{\frac{s}{2}}\int_{-\infty}^\infty \mathcal{F}_\pm\left(\begin{array}{c}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{array}\right)(\xi)\overline{\tilde{\mathcal{F}}_\pm[\beta_1(\lambda',.)]}d\xi d\lambda'$$
$$\int_0^{\frac{s}{2}}\int_{-\infty}^\infty \mathcal{F}_\pm\left(\begin{array}{c}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\end{array}\right)(\xi')\overline{\tilde{\mathcal{F}}_\pm[\beta_1(\lambda,.)](\xi')}d\xi' d\lambda$$

We may and shall restrict to the $+$ case and omit the subscript, and restrict both the ξ and ξ' integral to the range $(-\infty,0]$, the other case being similar but simpler. We then need to decompose each of the Fourier transforms $\mathcal{F}(...)$ etc into various constituents, i. e. write

$$\mathcal{F}\left(\begin{array}{c}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{array}\right)(\xi)=\langle\left(\begin{array}{c}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{array}\right),\phi(x,\xi)\rangle$$
$$+\langle\left(\begin{array}{c}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{array}\right),(1+r(-\xi))\underline{e}e^{-ix\xi}\rangle+\langle\left(\begin{array}{c}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{array}\right),(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle$$

We shall consider the contribution from the first and third term, the 2nd being treated similarly to the third. Moreover, performing the same decomposition for $\tilde{\mathcal{F}}[\beta_1(\lambda',.)]$ as well as $\mathcal{F}\left(\begin{array}{c}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\end{array}\right)(\xi')$, it is easy to see that we may restrict to the contribution from the third term, as the others are simpler. We commence with the following expression:

$$\sum_{\pm,\pm}\int_T^\infty t\int_t^\infty(\nu-1)(s)$$
$$\int_0^{\frac{s}{2}}\int_{-\infty}^0\langle\left(\begin{array}{c}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{array}\right),(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle\overline{\langle\beta_1(\lambda',.),(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle}d\xi d\lambda' \tag{5.50}$$
$$\int_0^{\frac{s}{2}}\int_{-\infty}^0\langle\left(\begin{array}{c}\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\end{array}\right),(e^{ix\xi'}-e^{-ix\xi'})\underline{e}\rangle\overline{\langle\beta_1(\lambda,.),(e^{ix\xi'}-e^{-ix\xi'})\underline{e}\rangle}d\xi' d\lambda$$

If we recapitulate the proof of SLDE for both bracket factors, we see that we may reduce to estimating

$$\sum_{\pm,\pm}\int_T^\infty t\int_t^\infty(\nu-1)(s)$$
$$\int_0^{\frac{s}{2}}\int_{-\infty}^0\langle\left(\begin{array}{c}P_{<a'}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a'}[|\tilde{U}^{(s)}|^2](\lambda',.)\\ -P_{<a'}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a'}[|\tilde{U}^{(s)}|^2](\lambda',.)\end{array}\right),(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle\overline{\langle\beta_1(\lambda',.),(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle}d\xi d\lambda'$$
$$\int_0^{\frac{s}{2}}\int_{-\infty}^0\langle\left(\begin{array}{c}P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda,.)\\ -P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda,.)\end{array}\right),(e^{ix\xi'}-e^{-ix\xi'})\underline{e}\rangle\overline{\langle\beta_1(\lambda,.),(e^{ix\xi'}-e^{-ix\xi'})\underline{e}\rangle}d\xi' d\lambda$$

where $a=\lambda^{-\frac{3}{4}+\epsilon_2}$, $a'=\lambda'^{-\frac{3}{4}}\lambda^{\epsilon_2}$, $\epsilon_2=\epsilon_2(\delta_2)$. Recalling the product representation of $\beta_1(\lambda',.)$ as in (5.47), we first reduce the frequency of the right hand integral factor of both $\beta_1(\lambda,.),\beta_1(\lambda',.)$ to size $>\lambda^{-\epsilon}$. This is achieved as in case (BC). For technical reasons we shall effect this by means of a sharp cutoff $\chi_{>\lambda^{-\epsilon}}(\xi)$ etc. Thus for example[79] we shall put

$$\beta_1(\lambda,x)=\left(\begin{array}{c}x\overline{\tilde{U}^{(s)}(\lambda,.)}\\ x\tilde{U}^{(s)}(\lambda,.)\end{array}\right)\times\chi_{>0}(x)\frac{e^{\pm i(s-\lambda)}}{(s-\lambda)^{\frac{3}{2}}}\int_{-\infty}^\infty e^{\frac{\pm(x-y)^2}{i(s-\lambda)}}g(y)dy,$$

where

$$g(y)=\phi_{<\lambda^{\frac{1}{10}}}(y)\int_{-\infty}^0\chi_{>\lambda^{-\epsilon}}(\xi)(e^{iy\xi}+e^{-iy\xi})\underline{e}[\frac{\mathcal{F}(\phi_{dis})(\xi)}{\xi}]d\xi$$

where $\chi_{>\lambda^{-\epsilon}}(\xi)$ is a Heavyside function. Of course we have

$$\frac{1}{\sqrt{s-\lambda}}\int_{-\infty}^\infty e^{\frac{\pm(x-y)^2}{i(s-\lambda)}}g(y)dy=\int_{-\infty}^\infty e^{\mp i(s-\lambda)\xi^2}e^{ix\xi}\hat{g}(\xi)d\xi$$

[79]The same argument applies to all terms of the list β_1.

Similar observations apply to $\beta_1(\lambda', .)$, for example

$$\beta_1(\lambda', x) = \begin{pmatrix} x\overline{\tilde{U}^{(s)}(\lambda', .)} \\ x\tilde{U}^{(s)}(\lambda', .) \end{pmatrix} \times \chi_{>0}(x)\frac{e^{\mp i(s-\lambda')}}{(s-\lambda')^{\frac{3}{2}}}\int_{-\infty}^{\infty} e^{\pm\frac{(x-y)^2}{i(s-\lambda)}}\tilde{g}(y)dy,$$

where

$$\tilde{g}(y) = \phi_{<\lambda^{\frac{1}{10}}}(y)\int_0^\infty \chi_{>\lambda^{-\epsilon}}(\xi')s(\xi')e^{iy\xi'}\underline{e}\frac{\mathcal{F}\phi_{dis}(\xi')}{(\xi'^2+1)^k\xi'}d\xi'$$

We now further specialize the frequency support of $g(y)$, $\tilde{g}(y)$, by including cutoffs $\chi_{I_i}(\xi)$, $\chi_{I_j}(\xi')$ corresponding to intervals $I_{i,j}$ of length $\lambda^{-\epsilon}$, i. e. introduce

$$g_i(y) = \phi_{<\lambda^{\frac{1}{10}}}(y)\int_{-\infty}^0 \chi_{>\lambda^{-\epsilon}}(\xi)\chi_{I_i}(\xi)(e^{iy\xi}+e^{-iy\xi})\underline{e}[\frac{\mathcal{F}(\phi_{dis})(\xi)}{\xi}]d\xi$$

$$\tilde{g}_j(y) = \phi_{<\lambda^{\frac{1}{10}}}(y)\int_0^\infty \chi_{>\lambda^{-\epsilon}}(\xi')s(\xi')e^{iy\xi'}\chi_j(\xi')\underline{e}\frac{\mathcal{F}\phi_{dis}(\xi')}{(\xi'^2+1)^k\xi'}d\xi'$$

Clearly if $|i-j| >> 1$ these functions have separated Fourier supports (of distance $\gtrsim \lambda^{-\epsilon}$) up to errors of order λ^{-N}, hence negligible. Now introduce $\beta_{1,i}(\lambda', .)$, $\beta_{1,j}(\lambda, .)$ exactly as above with $g(y)$, $\tilde{g}(y)$ replaced by $g_i(y)$, $\tilde{g}_j(y)$. It is easy to see that we can restrict both $|\xi|, |\xi'|$ to size $< \lambda^{\epsilon_3}$ for $\epsilon_3 = \epsilon_3(\delta_2)$, since otherwise one gains enough to overcome any losses in the proof of SLDE.

Case 1: $|i-j| >> 1$. Here we exploit integration by parts in s. Write

$$\beta_{1,i}(\lambda, x) = \begin{pmatrix} x\overline{\tilde{U}^{(s)}(\lambda, .)} \\ x\tilde{U}^{(s)}(\lambda, .) \end{pmatrix} \times \chi_{>0}(x)\frac{e^{\pm i(s-\lambda)}}{(s-\lambda)^{\frac{3}{2}}}\int_{-\infty}^{\infty} e^{\mp i(s-\lambda)\xi^2}e^{ix\xi}\widehat{g_i}(\xi)d\xi$$

and similarly for $\beta_{1,j}(\lambda', .)$. Then re-write

$$\sum_{\pm,\pm}\int_T^\infty t\int_t^\infty(\nu-1)(s)$$

$$\int_0^{\frac{s}{2}}\int_{-\infty}^0 \langle\begin{pmatrix} P_{<a'}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a'}[|\tilde{U}^{(s)}|^2](\lambda', .) \\ -P_{<a'}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a'}[|\tilde{U}^{(s)}|^2](\lambda', .) \end{pmatrix}, (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle\overline{\langle\beta_{1,j}(\lambda', .), (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle}d\xi d\lambda' \tag{5.51}$$

$$\int_0^{\frac{s}{2}}\int_{-\infty}^0 \langle\begin{pmatrix} P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda, .) \\ -P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda, .) \end{pmatrix}, (e^{ix\xi'}-e^{-ix\xi'})\underline{e}\rangle\overline{\langle\beta_{1,i}(\lambda, .), (e^{ix\xi'}-e^{-ix\xi'})\underline{e}\rangle}d\xi' d\lambda$$

as

$$\int_T^\infty t\int_t^\infty(\nu-1)(s)\int_{-\infty}^0\int_{-\infty}^0\int_0^\infty\int_0^\infty e^{is(\pm\xi^2\pm\xi'^2)}\Psi_{ij}(s,\lambda,\lambda',\xi,\xi')d\xi d\xi' d\lambda d\lambda' ds dt$$

switch the order of integration, integrate by parts in s, decouple the variables ξ, ξ' by means of discrete Fourier transform and proceed as in the proof of the SLDE. Choosing $\epsilon > 0$ small enough results in a gain in λ, even upon summing over i, j. This concludes Case 1.

Case 2: $i = j + O(1)$. First write

$$\beta_{1,i} = 2\sqrt{-1}\beta_{1,i}^a + \beta_{1,i}^b,$$

where

$$\beta_{1,i}^a = \lambda\begin{pmatrix} \overline{\partial_x\tilde{U}^{(s)}(\lambda, .)} \\ -\partial_x\tilde{U}^{(s)}(\lambda, .) \end{pmatrix} \times \chi_{>0}(x)\frac{e^{\pm i(s-\lambda)}}{(s-\lambda)^{\frac{3}{2}}}\int_{-\infty}^{\infty} e^{\frac{\pm(x-y)^2}{i(s-\lambda)}}g_i(y)dy$$

$$\beta_{1,i}^b = \begin{pmatrix} \overline{(x+2\lambda p)\tilde{U}^{(s)}(\lambda, .)} \\ (x+2\lambda p)\tilde{U}^{(s)}(\lambda, .) \end{pmatrix} \times \chi_{>0}(x)\frac{e^{\pm i(s-\lambda)}}{(s-\lambda)^{\frac{3}{2}}}\int_{-\infty}^{\infty} e^{\frac{\pm(x-y)^2}{i(s-\lambda)}}g_i(y)dy$$

From the proof of SLDE, recall that we use integration by parts to write

$$
\begin{aligned}
&\langle\beta_{1,i}^{(a)}(\lambda,.),(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle \\
&= i\lambda\xi\int_0^\infty(e^{ix\xi}+e^{-ix\xi})\langle\underline{e},\int_x^\infty\chi_{>0}(y)\begin{pmatrix}\overline{\partial_y\tilde{U}^{(s)}}(\lambda,y)\\ -\partial_y\tilde{U}^{(s)}(\lambda,y)\end{pmatrix}\times\frac{e^{\pm i(s-\lambda)}}{(s-\lambda)^{\frac{3}{2}}}\int_{-\infty}^\infty e^{\frac{\pm(y-z)^2}{i(s-\lambda)}}g_i(z)dzdy
\end{aligned}
\tag{5.52}
$$

Still following the proof of SLDE in the low-low case, we then use the free parametrix to write schematically

$$
\overline{\tilde{U}^{(s)}}(\lambda,y)=\int_0^\lambda\frac{1}{\sqrt{\lambda-\mu}}\int_{-\infty}^\infty e^{\frac{(y-z')^2}{i(\lambda-\mu)}}\partial_{z'}[|U|^4U(\mu,z')+VU(\mu,z')]dz'd\mu
$$

We then break this into the contributions from $\chi_{><\lambda^{\frac{1}{2}+\epsilon_3}}(\mu)\chi_{><(\lambda-\mu)^{\frac{1}{2}+\epsilon_3}}(|y|)|U|^4U(\mu,z')$, $\chi_{><\lambda-\lambda^{2\epsilon_3}}(\mu)\chi_{><\lambda^{\epsilon_3}}(z')VU(\mu,z')$. Indeed, from the argument in the proof of SLDE, it follows that if one $>$ sign is chosen in these cutoffs, the corresponding contribution leads to a small extra gain in λ, which then suffices to close, provided $\epsilon_3=\epsilon_3(\delta_2)$. Thus we now choose everywhere the $<$ sign, and substitute this into (5.52). Collecting the exponentials, we encounter the following phase function, just as in the proof of SLDE:

$$
\begin{aligned}
&e^{\frac{y^2}{i}[\frac{1}{\lambda-\mu}+\frac{1}{s-\lambda}]-\frac{2x}{i}[\frac{z'}{\lambda-\mu}+\frac{z}{s-\lambda}]}e^{\frac{z'^2}{i(\lambda-\mu)}+\frac{z^2}{i(s-\lambda)}} \\
&=e^{-i(y\sqrt{\frac{1}{\lambda-\mu}+\frac{1}{s-\lambda}}-[\frac{z'}{\lambda-\mu}+\frac{z}{s-\lambda}]\sqrt{\frac{1}{\lambda-\mu}+\frac{1}{s-\lambda}}^{-1})^2}e^{\frac{z'^2}{i(\lambda-\mu)}+\frac{z^2}{i(s-\lambda)}+i[\frac{z'}{\lambda-\mu}+\frac{z}{s-\lambda}]^2[\frac{1}{\lambda-\mu}+\frac{1}{s-\lambda}]^{-1}} \\
&:=e^{-i(y\sqrt{\frac{1}{\lambda-\mu}+\frac{1}{s-\lambda}}-y_1)^2}e^{iy_2}
\end{aligned}
$$

where we have

$$
|y_{1,2}|=|y_{1,2}(z,z',\lambda,\mu,s)|\lesssim\lambda^{\epsilon_3}
$$

on the support of the integrand in (5.52). Thus plugging this into (5.52) and omitting the integration in μ',μ for now, we obtain the expression

$$
\begin{aligned}
&\frac{1}{\sqrt{(s-\lambda)(\lambda-\mu)}}\int_{-\infty}^\infty\int_{-\infty}^\infty\int_0^\infty(e^{ix\xi}+e^{-ix\xi})\int_x^\infty e^{-i(y\sqrt{\frac{1}{\lambda-\mu}+\frac{1}{s-\lambda}}-y_1)^2}e^{iy_2}g_i(z)g_1(\mu,z')dydxdzdz' \\
&=\frac{[\frac{1}{s-\lambda}+\frac{1}{\lambda-\mu}]^{-1}}{\sqrt{(s-\lambda)(\lambda-\mu)}}\int_{-\infty}^\infty\int_{-\infty}^\infty e^{iy_2}\int_{-y_1}^\infty[e^{i\xi[\tilde{x}+y_1]\sqrt{\frac{1}{s-\lambda}+\frac{1}{\lambda-\mu}}^{-1}}+e^{-i\xi[\tilde{x}+y_1]\sqrt{\frac{1}{s-\lambda}+\frac{1}{\lambda-\mu}}^{-1}}]\int_{\tilde{x}}^\infty e^{i\rho^2}d\rho d\tilde{x} \\
&\qquad\phi_{<\lambda^{\frac{1}{10}}}(z)\int_{-\infty}^0\chi_{>\lambda^{-\epsilon}}(\eta)\chi_{I_i}(\eta)(e^{iz\eta}+e^{-iz\eta})\underline{e}[\frac{\mathcal{F}(\phi_{dis})(\eta)}{\eta}]d\eta g_1(z',\mu)dzdz'
\end{aligned}
$$

Now assume $I_i=[a_i,b_i]$ where $\min\{|a_i|,|b_i|\}\geq\lambda^{-\epsilon}$. Then integrate by parts in

$$
\begin{aligned}
&\int_{-\infty}^0\chi_{>\lambda^{-\epsilon}}(\eta)\chi_{I_i}(\eta)(e^{iz\eta}+e^{-iz\eta})\underline{e}[\frac{\mathcal{F}(\phi_{dis})(\eta)}{\eta}]d\eta \\
&=\frac{e^{iza_i}\frac{\mathcal{F}(\phi_{dis})(a_i)}{a_i}-e^{izb_i}\frac{\mathcal{F}(\phi_{dis})(b_i)}{b_i}}{iz}-\frac{1}{iz}\int_{-\infty}^0\chi_{>\lambda^{-\epsilon}}(\eta)\chi_{I_i}(\eta)(e^{iz\eta}-e^{-iz\eta})\underline{e}\partial_\eta[\frac{\mathcal{F}(\phi_{dis})(\eta)}{\eta}]d\eta
\end{aligned}
$$

Observe that

$$
||\phi_{<\lambda^{\frac{1}{10}}}(z)\frac{1}{iz}\int_{-\infty}^0\chi_{>\lambda^{-\epsilon}}(\eta)\chi_{I_i}(\eta)(e^{iz\eta}-e^{-iz\eta})\underline{e}\partial_\eta[\frac{\mathcal{F}(\phi_{dis})(\eta)}{\eta}]d\eta||_{L_z^1}\lesssim\log\lambda\lambda^{-\epsilon},
$$

whence this expression has a negligible contribution upon continuing the proof of SLDE and choosing $\epsilon_3<<\epsilon$. We further observe that the restriction $|\xi|<\lambda^{-\frac{3}{4}+\epsilon}$ as well as the restrictions on $|z|$, $|z'|$, λ and μ specified further above imply that

$$
|\partial_z[e^{iy_2}\int_{-y_1}^\infty[e^{i\xi[\tilde{x}+y_1]\sqrt{\frac{1}{s-\lambda}+\frac{1}{\lambda-\mu}}^{-1}}+e^{-i\xi[\tilde{x}+y_1]\sqrt{\frac{1}{s-\lambda}+\frac{1}{\lambda-\mu}}^{-1}}]\int_{\tilde{x}}^\infty e^{i\rho^2}d\rho d\tilde{x}]|\lesssim\lambda^{-\frac{1}{2}+\epsilon_3}
$$

$$
|[e^{iy_2}\int_{-y_1}^\infty[e^{i\xi[\tilde{x}+y_1]\sqrt{\frac{1}{s-\lambda}+\frac{1}{\lambda-\mu}}^{-1}}+e^{-i\xi[\tilde{x}+y_1]\sqrt{\frac{1}{s-\lambda}+\frac{1}{\lambda-\mu}}^{-1}}]\int_{\tilde{x}}^\infty e^{i\rho^2}d\rho d\tilde{x}]|\lesssim\lambda^{\epsilon_3}
$$

Thus for all intents and purposes we can replace the latter function by a constant as far as its dependence of z on $[-\lambda^{\frac{1}{10}}, \lambda^{\frac{1}{10}}]$ is concerned. But then one calculates

$$|\int_{-\infty}^{\infty} \phi_{<\lambda^{\frac{1}{10}}}(z) \frac{e^{iza_i}\frac{\mathcal{F}(\phi_{dis})(a_i)}{a_i} - e^{izb_i}\frac{\mathcal{F}(\phi_{dis})(b_i)}{b_i}}{iz} dz| \lesssim \lambda^{-\epsilon}$$

Putting everything after (5.52) together, we see that

$$|\langle \beta_{1,i}^{(a)}(\lambda,.), (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle| \lesssim \lambda\lambda^{-\frac{3}{4}+\epsilon_2}\lambda^{\frac{1}{2}}\log\lambda\lambda^{-\epsilon}(s-\lambda)^{-\frac{3}{2}},$$

which then yields

$$|\int_{-\infty}^{0}\langle\begin{pmatrix} P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda,.) \\ -P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda,.)\end{pmatrix}, (e^{ix\xi'} - e^{-ix\xi'})\underline{e}\rangle\overline{\langle\beta_{1,i}^{(a)}(\lambda,.), (e^{ix\xi'} - e^{-ix\xi'})\underline{e}\rangle} d\xi'|$$
$$\lesssim \lambda^{-1}\lambda^{-\frac{3}{4}+\epsilon_2}\lambda\lambda^{-\frac{3}{4}+\epsilon_2}\lambda^{\frac{1}{2}}\log\lambda\lambda^{-\epsilon}(s-\lambda)^{-\frac{3}{2}} \lesssim \log\lambda\lambda^{-1+2\epsilon_2-\epsilon}(s-\lambda)^{-\frac{3}{2}}$$

We shall choose $0 < \epsilon_2 << \epsilon$. Analogously to (5.52), we also need to estimate the contribution of

$$\langle\beta_{1,i}^{(b)}(\lambda,.), (e^{ix\xi} - e^{-ix\xi})\underline{e}\rangle$$

This, however, is more elementary, as we can estimate

$$||\langle(e^{ix\xi} - e^{-ix\xi})\underline{e}, \begin{pmatrix}\overline{(x+2\lambda p)\tilde{U}^{(s)}(\lambda,.)} \\ (x+2\lambda p)\tilde{U}^{(s)}(\lambda,.)\end{pmatrix} \times \chi_{>0}(x)\frac{e^{\pm i(s-\lambda)}}{(s-\lambda)^{\frac{3}{2}}}\int_{-\infty}^{\infty} e^{\frac{\pm(x-y)^2}{i(s-\lambda)}} g_i(y)dy||_{L^2_\xi}$$
$$\lesssim ||(x+2\lambda p)\tilde{U}^{(s)}(\lambda,.)||_{L^2_x}(s-\lambda)^{-\frac{3}{2}} \lesssim (s-\lambda)^{-\frac{3}{2}},$$

whence we get

$$|\int_{-\infty}^{0}\langle\begin{pmatrix} P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.) \\ -P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.)\end{pmatrix}, (e^{ix\xi'} - e^{-ix\xi'})\underline{e}\rangle\overline{\langle\beta_{1,i}^{(a)}(\lambda,.), (e^{ix\xi'} - e^{-ix\xi'})\underline{e}\rangle} d\xi'|$$
$$\lesssim \lambda^{-\frac{3}{2}}(s-\lambda)^{-\frac{3}{2}},$$

which then leads to an acceptable contribution. Now of course we eventually need to estimate the expression (5.51) under our current assumption $i = j + O(1)$, and then sum over i. One can replicate the preceding arguments for $\beta_{1,j}(\lambda',.)$ as long as $\lambda' > \lambda^{10\epsilon}$, say. But we can exclude the opposite case, since if $\lambda' \leq \lambda^{10\epsilon}$, we can restrict $|U(\lambda', x)$ to the range $|x| < \lambda^{20\epsilon}$, say, using pseudo-conformal almost conservation, and then we can restrict $e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\phi_{dis}$ to frequency $< s^{-10\epsilon}$, say, provided ϵ is small enough. Thus we can reduce this situation to the separated frequency case. Finally, if $\lambda' > \lambda^{10\epsilon}$, we get

$$|\int_{-\infty}^{0}\langle\begin{pmatrix} P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda,.) \\ -P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda,.)\end{pmatrix}, (e^{ix\xi'} - e^{-ix\xi'})\underline{e}\rangle\overline{\langle\beta_{1,i}^{(a)}(\lambda,.), (e^{ix\xi'} - e^{-ix\xi'})\underline{e}\rangle} d\xi'|$$
$$\times|\int_{-\infty}^{0}\langle\begin{pmatrix} P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.) \\ -P_{<a}[\chi_{>0}|\tilde{U}^{(s)}|^2]P_{<a}[|\tilde{U}^{(s)}|^2](\lambda',.)\end{pmatrix}, (e^{ix\xi'} - e^{-ix\xi'})\underline{e}\rangle\overline{\langle\beta_{1,j}^{(b)}(\lambda,.), (e^{ix\xi'} - e^{-ix\xi'})\underline{e}\rangle} d\xi'|$$
$$\lesssim \lambda^{-1}\lambda^{-\frac{3}{4}+\epsilon_2}\lambda\lambda^{-\frac{3}{4}+\epsilon_2}\lambda^{\frac{1}{2}}\log\lambda\lambda^{-\epsilon}(s-\lambda)^{-\frac{3}{2}}\lambda'^{-1}\lambda'^{-\frac{3}{4}}\lambda^{\epsilon_2}\lambda'\lambda'^{-\frac{3}{4}}\lambda^{\epsilon_2}\lambda'^{\frac{1}{2}}\log\lambda\lambda^{-\epsilon}(s-\lambda')^{-\frac{3}{2}}$$
$$\lesssim (\log\lambda)^2\lambda^{-1+4\epsilon_2-2\epsilon}\lambda'^{-1}(s-\lambda)^{-\frac{3}{2}}(s-\lambda')^{-\frac{3}{2}}$$

Upon summing over $O(\lambda^{\epsilon+\epsilon_3})$ indices $i = j + O(1)$, this leads to a small gain provided $\epsilon_2 + \epsilon_3 << \epsilon$, which we may arrange. This concludes the treatment of (5.50).

We next consider the contribution of the term

$$\sum_{\pm,\pm}\int_T^\infty t\int_t^\infty (\nu-1)(s)$$

$$\int_0^{\frac{s}{2}}\int_{-\infty}^0 \langle\begin{pmatrix} \chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.) \\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix},\phi(x,\xi)\rangle\overline{\langle\beta_1(\lambda',.),(e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle}d\xi d\lambda' \tag{5.53}$$

$$\int_0^{\frac{s}{2}}\int_{-\infty}^0 \langle\begin{pmatrix} \chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.) \\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda,.)\end{pmatrix},(e^{ix\xi'}-e^{-ix\xi'})\underline{e}\rangle\overline{\langle\beta_1(\lambda,.),(e^{ix\xi'}-e^{-ix\xi'})\underline{e}\rangle}d\xi' d\lambda$$

We observe that we may innocuously include cutoffs $\phi_{<>\lambda'\lambda^\epsilon}(|x|)$ simultaneously[80] in front of

$$\begin{pmatrix} \chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.) \\ -\chi_{>0}|\tilde{U}^{(s)}|^4(\lambda',.)\end{pmatrix},\ \beta_1(\lambda',.)$$

Including the cutoff $\phi_{>\lambda'\lambda^\epsilon}(|x|)$ clearly leads to the desired extra gain, while including the cutoff $\phi_{<\lambda'\lambda^\epsilon}(|x|)$ allows us to restrict $e^{-i(s-\lambda')\mathcal{H}^*}(\mathcal{H}^*)^{-k}\psi_{dis}$ in $\beta_{1,\lambda'}(\lambda',.)$ to small frequency. As we we can always restrict the frequency of $e^{-i(s-\lambda)\mathcal{H}}\phi_{dis}$ in $\beta_1(\lambda,.)$ away from zero, we have then achieved frequency separation and can argue as before in case (BC). This concludes the case (CC) provided we have $\max\{\lambda,\lambda'\}<\frac{s}{2}$. The case $\max\{\lambda,\lambda'\}\geq\frac{s}{2}$ is more elementary and omitted. We are now done with the proof of Lemma 5.6, whence also the Lemma before it. □

We now continue with the proof of Proposition 5.4; note that the expressions

$$\int_T^\infty t\int_t^\infty (\nu-1)^a(s)\langle\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}(s,.),\phi\rangle dsdt,\ \int_T^\infty t\int_t^\infty \dot{\nu}(s)\langle\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis}(s,.),\phi\rangle dsdt,\ a\geq 1$$

can be treated exactly like above, using Lemma 5.6 and the relation (3.24). Thus up to terms which can either be estimated using Lemma 5.6 or else can even be absolutely integrated, there is only one potentially troublesome expression, namely

$$\int_T^\infty t\int_t^\infty\langle\tilde{U}^2-\overline{\tilde{U}}^2,\phi\rangle dsdt$$

where ϕ is an even time-independent Schwartz function. We handle this by the following Lemma, which hinges on a *symplectic structure*:

Lemma 5.7. *Let* $\Gamma\in A^{(n)}_{[0,T)}$, $\Gamma=\{\begin{pmatrix}\tilde{U}\\ \bar{\tilde{U}}\end{pmatrix}_{dis},\ldots\}$. *Then, for* $\tilde{T}\leq T$ *and* ϕ *an even Schwartz function, we have*

$$\int_{\tilde{T}}^T t\int_t^T\langle\tilde{U}_{dis}^2(s,.)-\overline{\tilde{U}_{dis}}^2(s,.),\phi\rangle dsdt\lesssim\tilde{T}^{-\frac{1}{2}+\delta_1},$$

provided δ_1 *is large enough in relation to* δ_2,δ_3. *Similarly, we have the bound*

$$\int_{\tilde{T}}^T t^2\langle\tilde{U}_{dis}^2(t,.)-\overline{\tilde{U}_{dis}}^2(t,.),\phi\rangle dt\lesssim\tilde{T}^{-\frac{1}{2}+\delta_1},$$

Both bounds are uniform in T.

Proof. It relies on identifying a special cancellation in this expression. We treat the first expression in detail, the 2nd following the same reasoning. Also, we may put $T=\infty$. To begin with, write

$$\begin{pmatrix}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{pmatrix}_{dis}=\sum_\pm\int_{-\infty}^\infty e_\pm(x,\xi)\mathcal{F}_\pm\begin{pmatrix}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{pmatrix}(\xi)d\xi \tag{5.54}$$

This gets substituted into

$$\langle(\tilde{U}^{(s)}_{dis})^2(s,.)-(\overline{\tilde{U}^{(s)}_{dis}})^2(s,.),\phi\rangle=\langle\begin{pmatrix}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{pmatrix}^t_{dis}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\tilde{U}^{(s)}\\ \overline{\tilde{U}^{(s)}}\end{pmatrix}_{dis},\phi\rangle \tag{5.55}$$

[80]I. e. either both have the < or the > subscript.

The key here is that

$$\langle e_+(x,\xi)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e_-(x,\xi'), \phi(x)\rangle = \langle e_+(x,\xi)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e_+(x,\xi'), \phi(x)\rangle,$$

which is easily seen to vanish for $\xi = \xi'$. Moreover, this also vanishes for $\xi = -\xi'$, since then it is the inner product of an odd and an even function (use that $e_\pm(x,-\xi) = e_\pm(-x,\xi)$). To proceed, we now substitute the Duhamel expression for $\begin{pmatrix} \tilde{U}^{(s)} \\ \overline{\tilde{U}^{(s)}} \end{pmatrix}_{dis}$ into (5.55). We shall then treat the most difficult term which results when we substitute the non-local source term for both factors, i. e. the expression

$$\sum_{\pm,\pm}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_T^{\infty} t\int_t^{\infty}\langle e_\pm(x,\xi)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e_\pm(x,\xi'), \phi(x)\rangle$$
$$\int_0^s e^{\mp i(s-\lambda)(\xi^2+1)}\mathcal{F}_\pm\begin{pmatrix} |\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.) \end{pmatrix}(\xi)d\lambda$$
$$\int_0^s e^{\mp i(s-\lambda')(\xi'^2+1)}\mathcal{F}_\pm\begin{pmatrix} |\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda',.) \\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda',.) \end{pmatrix}(\xi')d\lambda' d\xi d\xi' ds dt$$

The remaining (local) source terms are handled by the exact same method but much easier, hence omitted. Then we focus on the most difficult case when the s-phases cancel each other, i. e. when there is a + and a − sign. In order to render the Fourier transforms explicit, we write as usual

$$\begin{pmatrix} |\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.) \end{pmatrix} = \chi_{>0}(x)\begin{pmatrix} |\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.) \end{pmatrix} + \chi_{\leq 0}(x)\begin{pmatrix} |\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.) \end{pmatrix},$$

with a similar expression for the 2nd factor in λ'. W. l. o. g. we shall then include the $\chi_{>0}(x)$-cutoff in both cases. Then we subdivide the (ξ,ξ')-plane into the four standard quadrants. If (ξ,ξ') is in the first or third quadrant, observe that

$$\frac{\langle \frac{e_+(x,\xi)}{\xi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{e_+(x,\xi')}{\xi'}, \phi(x)\rangle}{\xi-\xi'} = \frac{\langle [\frac{e_+(x,\xi)}{\xi} - \frac{e_+(x,\xi')}{\xi'}]\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{e_+(x,\xi')}{\xi'}, \phi(x)\rangle}{\xi-\xi'},$$

whence this is smooth and bounded with bounded derivatives in the interior of these quadrants, and continuous up to the boundary. If (ξ,ξ') is in one of the other quadrants, we have

$$\frac{\langle \frac{e_+(x,\xi)}{\xi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{e_+(x,\xi')}{\xi'}, \phi(x)\rangle}{\xi+\xi'} = \frac{\langle [\frac{e_+(x,\xi)}{\xi} - \frac{e_+(x,-\xi')}{-\xi'}]\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\frac{e_+(x,\xi')}{\xi'}, \phi(x)\rangle}{\xi+\xi'},$$

whence the same comment applies. Now let (ξ,ξ') be in the third quadrant. Then we can write

$$\mathcal{F}_+\begin{pmatrix} \chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.) \end{pmatrix}(\xi)$$
$$= \langle\begin{pmatrix} \chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.) \end{pmatrix}, (e^{ix\xi}-e^{-ix\xi})\underline{e} + (1+r(-\xi))e^{-ix\xi}\underline{e} + \phi(x,\xi)\rangle,$$

with a similar expression for $\mathcal{F}_-\begin{pmatrix} \chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda',.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda',.) \end{pmatrix}(\xi')$, where $\underline{e}$ gets replaced by $\sigma_1\underline{e}$. We shall treat in detail the contribution of $(e^{ix\xi}-e^{-ix\xi})\underline{e}$, the other contributions being treated similarly. As usual we need to distinguish between different frequency ranges: first assume $\max\{|\xi|,|\xi'|\} > s^{\epsilon(\delta_2)}$, for suitable $\epsilon(\delta_2)$.

For example assume $|\xi| > s^{\epsilon(\delta_2)}$, effected by means of a smooth cutoff $\phi_{>s^\epsilon}(\xi)$. Observe that then

$$\langle\chi_{>0}(x)\begin{pmatrix} |\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}, (e^{ix\xi}-e^{-ix\xi})\underline{e}\rangle + = O(\frac{1}{s^\epsilon})[\langle\delta_0(x)\begin{pmatrix} |\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}, (e^{ix\xi}+e^{-ix\xi})\underline{e}\rangle$$
$$+ \langle\chi_{>0}(x)\partial_x\begin{pmatrix} |\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}, (e^{ix\xi}+e^{-ix\xi})\underline{e}\rangle]$$

For the boundary term, integrate by parts in ξ in order to score arbitrary gains in s. For the 2nd term, keep integrating by parts in x until a boundary term results or else enough powers of s are gained.
Thus we may now include smooth cutoffs $\phi_{<s^\epsilon}(\xi)$, $\phi_{<s^\epsilon}(\xi')$. Commence with the case $\max\{\lambda,\lambda'\} < \frac{s}{2}$. We perform integrations by parts in ξ, ξ', and obtain the following list of integrals

$$A = \int_{-\infty}^0\int_{-\infty}^0\int_T^\infty t\int_t^\infty \phi_{<s^\epsilon}(\xi)\phi_{<s^\epsilon}(\xi')\langle\frac{e(x,\xi)}{\xi}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\frac{e(x,\xi')}{\xi'},\phi(x)\rangle$$
$$\int_0^{\frac{s}{2}}\frac{1}{s-\lambda}e^{-i(s-\lambda)(\xi^2+1)}\partial_\xi\mathcal{F}_+\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}(\xi)d\lambda$$
$$\int_0^{\frac{s}{2}}\frac{1}{s-\lambda'}e^{+i(s-\lambda)(\xi'^2+1)}\partial_{\xi'}\mathcal{F}_-\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda',.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda',.)\end{pmatrix}(\xi')d\lambda' ds dt d\xi d\xi'$$

$$B = \int_{-\infty}^0\int_{-\infty}^0\int_T^\infty t\int_t^\infty \phi_{<s^\epsilon}(\xi)\phi_{<s^\epsilon}(\xi')\langle\partial_\xi[\frac{e(x,\xi)}{\xi}]\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\frac{e(x,\xi')}{\xi'},\phi(x)\rangle$$
$$\int_0^{\frac{s}{2}}\frac{1}{s-\lambda}e^{-i(s-\lambda)(\xi^2+1)}\mathcal{F}_+\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}(\xi)d\lambda$$
$$\int_0^{\frac{s}{2}}\frac{1}{s-\lambda'}e^{+i(s-\lambda')(\xi'^2+1)}\partial_{\xi'}\mathcal{F}_-\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda',.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda',.)\end{pmatrix}(\xi')d\lambda' ds dt d\xi d\xi'$$

$$C = \int_{-\infty}^0\int_{-\infty}^0\int_T^\infty t\int_t^\infty \phi_{<s^\epsilon}(\xi)\phi_{<s^\epsilon}(\xi')\langle\frac{e(x,\xi)}{\xi}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\partial_{\xi'}[\frac{e(x,\xi')}{\xi'}],\phi(x)\rangle$$
$$\int_0^{\frac{s}{2}}\frac{1}{s-\lambda}e^{-i(s-\lambda)(\xi^2+1)}\partial_\xi\mathcal{F}_+\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}(\xi)d\lambda$$
$$\int_0^{\frac{s}{2}}\frac{1}{s-\lambda'}e^{+i(s-\lambda')(\xi'^2+1)}\mathcal{F}_-\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda',.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda',.)\end{pmatrix}(\xi')d\lambda' ds dt d\xi d\xi'$$

$$D = \int_{-\infty}^0\int_{-\infty}^0\int_T^\infty t\int_t^\infty \phi_{<s^\epsilon}(\xi)\phi_{<s^\epsilon}(\xi')\langle\partial_\xi[\frac{e(x,\xi)}{\xi}]\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\partial_{\xi'}[\frac{e(x,\xi')}{\xi'}],\phi(x)\rangle$$
$$\int_0^{\frac{s}{2}}\frac{1}{s-\lambda}e^{-i(s-\lambda)(\xi^2+1)}\mathcal{F}_+\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}(\xi)d\lambda$$
$$\int_0^{\frac{s}{2}}\frac{1}{s-\lambda'}e^{+i(s-\lambda')(\xi'^2+1)}\mathcal{F}_-\begin{pmatrix}\chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda',.) \\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda',.)\end{pmatrix}(\xi')d\lambda' ds dt d\xi d\xi'$$

We don't worry about the case when ∂_ξ or $\partial_{\xi'}$ falls on one of the cutoffs since then we have either $|\xi| \sim s^\epsilon$ or $|\xi'| \sim s^\epsilon$, which case is treated as before. Observe that we can write

$$\mathcal{F}_+\begin{pmatrix}|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}(\xi) = \xi\int_0^1\partial_\xi\mathcal{F}_+\begin{pmatrix}|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.) \\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{pmatrix}(\alpha\xi)d\alpha$$

Of course a similar identity applies to $\mathcal{F}_-(...)$, whence we reduce to estimating expressions of the form

$$\begin{aligned}\int_{-\infty}^0\int_{-\infty}^0\int_T^\infty t\int_t^\infty &\phi_{<s^\epsilon}(\xi)\phi_{<s^\epsilon}(\xi')\xi g(\xi,\xi')\\ &\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda}e^{-i(s-\lambda)(\xi^2+1)}\partial_\xi\mathcal{F}_+\left(\begin{array}{c}|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.)\\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{array}\right)(\xi)d\lambda\\ &\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda'}e^{+i(s-\lambda')(\xi'^2+1)}\partial_{\xi'}\mathcal{F}_-\left(\begin{array}{c}|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.)\\ -|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda',.)\end{array}\right)(\xi')d\lambda'd\xi d\xi',\end{aligned}\tag{5.56}$$

where the function $g(\xi,\xi')$ is smooth and bounded in the interior of the third quadrant as well as continuous up to the boundary. Expand as usual

$$\partial_\xi\mathcal{F}_+\left(\begin{array}{c}\chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.)\\ -\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{array}\right)(\xi)=\langle\left(\begin{array}{c}x\chi_{>0}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.)\\ -x\chi_{>0}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{array}\right),(e^{ix\xi}+e^{-ix\xi})\underline{e}\rangle+\dots$$

where $\dots$ represent terms that can be treated similarly. Now assume that we localize x to dyadic range $|x|\sim 2^k$, $k\geq 0$. If then we have $|\xi|>\max\{s^{-\frac{1}{2+}},s^{-1-2^k}\}$, effected by means of a smooth cutoff, we obtain arbitrary gains in s by integration by parts in ξ. Thus we shall now include a localizer $\phi_{<\max\{s^{-\frac{1}{2+}},s^{-1-2^k}\}}(\xi)$ upon localizing x to dyadic range $|x|\sim 2^k$, i. e. we reduce to considering

$$\begin{aligned}\sum_k\int_{-\infty}^0\int_{-\infty}^0\int_T^\infty t\int_t^\infty &\phi_{<\max\{s^{-\frac{1}{2+}},s^{-1-2^k}\}}(\xi)\phi_{<s^\epsilon}(\xi)\phi_{<s^\epsilon}(\xi')\xi g(\xi,\xi')\\ &\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda}e^{-i(s-\lambda)(\xi^2+1)}\langle\left(\begin{array}{c}\phi_{\sim 2^k}(x)x|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.)\\ -\phi_{\sim 2^k}(x)x|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{array}\right),(e^{ix\xi}+e^{-ix\xi})\underline{e}\rangle d\lambda\\ &\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda'}e^{+i(s-\lambda')(\xi'^2+1)}\langle\left(\begin{array}{c}x|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.)\\ -x|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{array}\right),(e^{ix\xi'}+e^{-ix\xi'})\sigma_1\underline{e}\rangle d\lambda'd\xi d\xi',\end{aligned}\tag{5.57}$$

Note that summing over k will amount to an extra $\log s$ at most, whence we shall safely discard this summation. Our strategy shall be to perform an integration by parts in $\langle\left(\begin{array}{c}\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,.)\\ -\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{U}^{(s)}}(\lambda,.)\end{array}\right),(e^{ix\xi}+e^{-ix\xi})\underline{e}\rangle$.
For this to be useful, though, we need to achieve some preliminary reductions in the last factor $\tilde{U}^{(s)}$, just as in the proof of the SLDE. Recall that we can write

$$\tilde{U}^{(s)}(\lambda,.)=\int_0^\lambda\sqrt{\lambda-\mu}^{-1}\int_{-\infty}^\infty e^{\frac{(y-z)^2}{i(\lambda-\mu)}}g(\mu,z)dzd\mu$$

where $g(\mu,z)=|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\mu,z)+\dots$. Decompose

$$|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\mu,z)=\chi_{<2^k}(\mu)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\mu,z)+\chi_{\geq 2^k}(\mu)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\mu,z)$$

Then observe that

$$||\int_0^\lambda\sqrt{\lambda-\mu}^{-1}\int_{-\infty}^\infty e^{\frac{(y-z)^2}{i(\lambda-\mu)}}\chi_{\geq 2^k}(\mu)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\mu,z)dz||_{L_y^2}\lesssim\int_{2^k}^\lambda\mu^{-2}d\mu\lesssim 2^{-k}$$

Similarly, we have

$$\begin{aligned}&||\int_0^\lambda\sqrt{\lambda-\mu}^{-1}\int_{-\infty}^\infty e^{\frac{(y-z)^2}{i(\lambda-\mu)}}\chi_{<2^k}(\mu)\chi_{\geq 2^{k-10}}(z)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\mu,z)dz||_{L_y^2}\\ &\lesssim\int_0^\lambda||\chi_{\geq s^k}(z)\frac{2i\mu\partial_z\tilde{U}^{(s)}(\mu,z)-C\tilde{U}^{(s)}(\mu,z)}{z}||_{L_z^2}\mu^{-2}d\mu\lesssim\log\lambda 2^{-k}\end{aligned}$$

One obtains similar estimates if one substitutes the remaining local terms in $g(\mu,z)$, localized to $|z|>2^{k-10}$, in the preceding integral. Thus if we substitute

$$V^{(s)}(\lambda,y):=\int_0^\lambda \sqrt{\lambda-\mu}^{-1}\int_{-\infty}^{\infty} e^{\frac{(y-z)^2}{i(\lambda-\mu)}}[\chi_{\geq 2^k}(\mu)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\mu,z)$$
$$+\chi_{<2^k}(\mu)\chi_{\geq 2^{k-10}}(z)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\mu,z)+\chi_{\geq 2^{k-10}}(z)...]dzd\mu$$

instead of $\tilde{U}^{(s)}$ for the last factor in $|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\lambda,y)$, we get

$$||\phi_{\sim 2^k}(x)x|\tilde{U}^{(s)}|^4V^{(s)}(\lambda,x)||_{L^1_x}\lesssim \lambda^{-\frac{3}{2}}$$

We now show how this suffices to control

$$\int_{-\infty}^0\int_{-\infty}^0\int_T^\infty t\int_t^\infty \phi_{<s^\epsilon}(\xi)\phi_{<s^\epsilon}(\xi')\phi_{<\max\{s^{-\frac{1}{2+}},s^{-1-}2^k\}}(\xi)\xi g(\xi,\xi')$$
$$\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda}e^{-i(s-\lambda)(\xi^2+1)}\langle\begin{pmatrix} x\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4V^{(s)}\lambda,x) \\ -x\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\overline{V^{(s)}}(\lambda,.)\end{pmatrix},(e^{ix\xi}+e^{-ix\xi})\underline{e}\rangle d\lambda$$
$$\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda'}e^{+i(s-\lambda)(\xi'^2+1)}\langle\begin{pmatrix} x|\tilde{U}^{(s)}|^4\tilde{V}^{(s)}(\lambda',.) \\ -x|\tilde{U}^{(s)}|^4\overline{\tilde{V}^{(s)}}(\lambda',.)\end{pmatrix},(e^{ix\xi'}+e^{-ix\xi'})\sigma_1\underline{e}\rangle d\lambda' d\xi d\xi',$$

where $\tilde{V}^{(s)}(\lambda',.)$ is defined analogously. Thus we can estimate this by

$$\int_T^\infty t\int_t^\infty \max\{s^{-\frac{1}{2+}},s^{-1-}2^k\}\int_0^{\frac{s}{2}}\frac{1}{(s-\lambda)^{\frac{3}{2}}}||\begin{pmatrix} x\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4V^{(s)}\lambda,x) \\ -x\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\overline{V^{(s)}}(\lambda,.)\end{pmatrix}||_{L^1_x}d\lambda$$
$$\int_0^{\frac{s}{2}}\frac{1}{(s-\lambda')^{\frac{3}{2}}}||\begin{pmatrix} x|\tilde{U}^{(s)}|^4\tilde{V}^{(s)}(\lambda',.) \\ -x|\tilde{U}^{(s)}|^4\overline{\tilde{V}^{(s)}}(\lambda',.)\end{pmatrix}||_{L^1_x}d\lambda'\lesssim\int_T^\infty t\int_t^\infty s^{-\frac{1}{2+}}s^{-3}dsdt\lesssim T^{-\frac{1}{2+}}$$

We are exploiting here the pseudo-conformal almost conservation, which implies that

$$||x\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4V^{(s)}\lambda,x)||_{L^1_x}\lesssim\frac{\lambda}{2^k}\lambda^{-\frac{3}{2}}$$

Thus we now replace at least one of $\tilde{U}^{(s)}(\lambda,.)$, $\tilde{U}^{(s)}(\lambda',.)$ by

$$\tilde{W}^{(s)}(\lambda,.):=\int_{-\infty}^\infty\int_0^\lambda\frac{1}{\sqrt{\lambda-\mu}}e^{\frac{(y-z)^2}{i(\lambda-\mu)}}[\chi_{<2^k}(\mu)\chi_{<2^{k-10}}(z)|\tilde{U}^{(s)}|^4\tilde{U}^{(s)}(\mu,z)+\ldots]d\mu dz\,\text{etc}$$

Then we decouple the ξ,ξ' variables in (5.57), which can be achieved by means of discrete Fourier expansion:

$$\phi_{<s^\epsilon}(|\xi|)\phi_{<s^\epsilon}(|\xi'|)\xi g(\xi,\xi')=\xi\sum_{n,m\in s^{-\epsilon}\mathbf{Z}}a_{nm}e^{in\xi+im\xi'},\ |a_{nm}|\lesssim[s^\epsilon|n|+s^\epsilon|m|]^{-N}$$

Consider the case when we replace the fifth $\tilde{U}^{(s)}(\lambda,.)$ by $\tilde{W}^{(s)}(\lambda,.)$. We are thus led to estimating contributions of the form

$$\int_{-\infty}^0\xi\phi_{<s^\epsilon}(\xi)e^{i(s-\lambda)(\xi^2+1)}\langle\chi_{>0}(x)\phi_{\sim 2^k}(x)x|\tilde{U}^{(s)}|^4\tilde{W}^{(s)}(\lambda,.),(e^{i(x+n)\xi}+e^{-i(x-n)\xi})\underline{e}\rangle d\xi$$

We shall put $n=0$ since the other cases are dealt with similarly. We treat here the contribution of $e^{-ix\xi}$, the one of $e^{+ix\xi}$ being treated similarly. Switch the order of integration in this, and introduce the new variable $\tilde{\xi}:=\sqrt{s-\lambda}\xi+\frac{x}{2\sqrt{s-\lambda}}$. Then we can rewrite the preceding expression as

$$\frac{1}{\sqrt{s-\lambda}}\int_0^\infty\int_{-\infty}^{\frac{x}{2\sqrt{s-\lambda}}}\phi_{<s^\epsilon}(\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda}}}{\sqrt{s-\lambda}})\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda}}}{\sqrt{s-\lambda}}e^{i\tilde{\xi}^2}e^{\frac{x^2}{4i(s-\lambda)}}x\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\tilde{W}^{(s)}(\lambda,.)d\tilde{\xi}dx$$

Now perform an integration by parts in the x-variable, and replace the preceding by the sum of multiples of the following expressions(as well as equivalent terms):

(5.58)
$$\frac{1}{\sqrt{s-\lambda}}\int_0^\infty\int_{-\infty}^{\frac{x}{2\sqrt{s-\lambda}}}\partial_x[\phi_{<s^\epsilon}(\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda}}}{\sqrt{s-\lambda}})\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda}}}{\sqrt{s-\lambda}}]e^{i\tilde{\xi}^2}d\tilde{\xi}x\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4[\int_x^\infty e^{\frac{y^2}{4i(s-\lambda)}}\tilde{W}^{(s)}(\lambda,y)dy]dx$$

$$\frac{1}{\sqrt{s-\lambda}}\int_0^\infty\int_{-\infty}^{\frac{x}{2\sqrt{s-\lambda}}}[\phi_{<s^\epsilon}(\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda}}}{\sqrt{s-\lambda}})\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda}}}{\sqrt{s-\lambda}}]e^{i\tilde{\xi}^2}d\tilde{\xi}\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4[\int_x^\infty e^{\frac{y^2}{4i(s-\lambda)}}\tilde{W}^{(s)}(\lambda,y)dy]dx \tag{5.59}$$

$$\begin{aligned}&\frac{1}{\sqrt{s-\lambda}}\int_0^\infty\int_{-\infty}^{\frac{x}{2\sqrt{s-\lambda}}}[\phi_{<s^\epsilon}(\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda}}}{\sqrt{s-\lambda}})\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda}}}{\sqrt{s-\lambda}}]e^{i\tilde{\xi}^2}d\tilde{\xi}x\phi_{\sim 2^k}(x)\partial_x[|\tilde{U}^{(s)}|^4(\lambda,x)]\\&\qquad[\int_x^\infty e^{\frac{y^2}{4i(s-\lambda)}}\tilde{W}^{(s)}(\lambda,y)dy]dx\end{aligned} \tag{5.60}$$

Now write
$$\tilde{W}^{(s)}(\lambda,y)=\int_{-\infty}^\infty\int_0^\lambda\frac{1}{\sqrt{\lambda-\mu}}e^{\frac{(y-z)^2}{i(\lambda-\mu)}}g(\mu,z)d\mu dz,$$
where we have $|z|<2^{k-10}$ on the support of $g(\mu,z)$. Thus we obtain
$$\begin{aligned}&\int_x^\infty e^{\frac{y^2}{4i(s-\lambda)}}\tilde{W}^{(s)}(\lambda,y)dy=\int_{-\infty}^\infty\int_x^\infty\int_0^\lambda\frac{1}{\sqrt{\lambda-\mu}}e^{\frac{y^2}{4i(s-\lambda)}}e^{\frac{(y-z)^2}{i(\lambda-\mu)}}g(\mu,z)d\mu dydz\\&=\int_{-\infty}^\infty\int_0^\lambda O(\frac{1}{\sqrt{\lambda-\mu}\sqrt{\frac{1}{4(s-\lambda)}+\frac{1}{\lambda-\mu}}}\frac{1}{x\sqrt{\frac{1}{4(s-\lambda)}+\frac{1}{\lambda-\mu}}-\frac{z}{(\lambda-\mu)\sqrt{\frac{1}{4(s-\lambda)}+\frac{1}{\lambda-\mu}}}})g(\mu,z)d\mu\phi_{<2^{k-10}}(|z|)dz,\end{aligned}$$
which is seen for $x\sim 2^k$ to be of order $(\frac{x}{\sqrt{\lambda}})^{-1}$, upon using the definition of $g(\mu,z)$. Now plug this into (5.58). For example, we get
$$\begin{aligned}&|\frac{1}{\sqrt{s-\lambda}}\int_0^\infty\int_{-\infty}^{\frac{x}{2\sqrt{s-\lambda}}}\frac{1}{s-\lambda}[\phi'_{<s^\epsilon}(\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda}}}{\sqrt{s-\lambda}})\frac{x}{2(s-\lambda)}e^{i\tilde{\xi}^2}d\tilde{\xi}x\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\\&\qquad[\int_x^\infty e^{\frac{y^2}{4i(s-\lambda)}}\tilde{W}^{(s)}(\lambda,y)dy]dx|\\&\lesssim\frac{1}{\sqrt{s-\lambda}}\frac{1}{(s-\lambda)^2}\sqrt{\lambda}\lambda^{\epsilon(\delta_2)}\end{aligned}$$
Observe that we are using pseudo-conformal almost conservation here. Integrating over $\lambda<\frac{s}{2}$ results in the upper bound $\lesssim s^{-1}$. The remaining contributions to (5.58) (Leibnitz rule) are treated similarly, as is the contribution of (5.59). Now consider (5.60). Here we invoke the same trick as in the proof of SLDE:
$$\partial_x[|\tilde{U}^{(s)}|^2(\lambda,x)]=\frac{1}{i\lambda}[C\tilde{U}^{(s)}\overline{\tilde{U}^{(s)}}(\lambda,x)-\tilde{U}^{(s)}\overline{C\tilde{U}^{(s)}}(\lambda,x)]$$
For example, we can estimate
$$\begin{aligned}&|\frac{1}{\sqrt{s-\lambda}}\int_0^\infty\int_{-\infty}^{\frac{x}{2\sqrt{s-\lambda}}}[\phi_{<s^\epsilon}(\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda}}}{\sqrt{s-\lambda}})\frac{x}{(s-\lambda)}e^{i\tilde{\xi}^2}d\tilde{\xi}x\phi_{\sim 2^k}(x)\partial_x[|\tilde{U}^{(s)}|^4(\lambda,x)]\\&\qquad[\int_x^\infty e^{\frac{y^2}{4i(s-\lambda)}}\tilde{W}^{(s)}(\lambda,y)dy]dx|\\&\lesssim|\frac{1}{\sqrt{s-\lambda}}\int_0^\infty\int_{-\infty}^{\frac{x}{2\sqrt{s-\lambda}}}[\phi_{<s^\epsilon}(\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda}}}{\sqrt{s-\lambda}})\frac{1}{(s-\lambda)}e^{i\tilde{\xi}^2}d\tilde{\xi}\phi_{\sim 2^k}(x)\frac{1}{\lambda}|C\tilde{U}^{(s)}||x\tilde{U}^{(s)}||\tilde{U}^{(s)}|^2(\lambda,x)]\sqrt{\lambda}d\tilde{\xi}dx\\&\lesssim(s-\lambda)^{-\frac{3}{2}}\sqrt{\lambda}^{-1}\lambda^{\epsilon(\delta_2)},\end{aligned}$$

which upon integration over $\lambda < \frac{s}{2}$ again results in the estimate s^{-1+}. The 2nd contribution to (5.60) is treated similarly.

Keeping in mind that we need to eventually estimate (5.57), we next consider the expression

$$\int_{-\infty}^{0}\phi_{<s^{\epsilon}}(\xi')e^{i(s-\lambda)(\xi'^2+1)}\langle\chi_{>0}(x)\phi_{\sim 2^k}(x)x|\tilde{U}^{(s)}|^4\tilde{W}^{(s)}(\lambda',x),(e^{ix\xi'}+e^{-ix\xi'})\underline{e}\rangle d\xi'$$

One proceeds analogously and obtains expressions as in (5.58), (5.59), (5.60) but without the factor $\frac{\tilde{\xi}'-\frac{x}{2\sqrt{s-\lambda'}}}{\sqrt{s-\lambda'}}$. One then has to argue somewhat differently for the expression

$$\frac{1}{\sqrt{s-\lambda'}}\int_0^{\infty}\int_{-\infty}^{\frac{x}{2\sqrt{s-\lambda'}}}\phi_{<s^{\epsilon}}(\frac{\tilde{\xi}-\frac{x}{2\sqrt{s-\lambda'}}}{\sqrt{s-\lambda'}})e^{i\tilde{\xi}^2}d\tilde{\xi}\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4(\lambda',x)[\int_x^{\infty}e^{\frac{y^2}{4i(s-\lambda')}}\tilde{W}^{(s)}(\lambda',y)dy]dx$$

Here we use that

$$|\tilde{U}^{(s)}|(\lambda',x)[\frac{\langle x\rangle^{\frac12}}{\sqrt{\lambda'}}]^{-1}\lesssim\sqrt{\lambda'}\lambda'^{-1+\epsilon(\delta_2)},$$

whence we can bound the above expression by

$$\frac{1}{\sqrt{s-\lambda'}}\lambda'^{-\frac32+\epsilon(\delta_2)}$$

Integrating over $\lambda' < \frac{s}{2}$ results in the upper bound $\sqrt{s}^{-1+\epsilon(\delta_2)}$. Combining all these estimates, we now obtain

$$\begin{aligned}|\sum_k\int_{-\infty}^{0}\int_{-\infty}^{0}\int_T^{\infty}t\int_t^{\infty}&\phi_{<s^{\epsilon}}(\xi)\phi_{<s^{\epsilon}}(\xi')\xi g(\xi,\xi')\\ &\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda}e^{-i(s-\lambda)(\xi^2+1)}\langle\begin{pmatrix}\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\tilde{W}^{(s)}(\lambda,.)\\ -\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{W}^{(s)}}(\lambda,.)\end{pmatrix},(e^{ix\xi}+e^{-ix\xi})\underline{e}\rangle d\lambda\\ &\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda'}e^{+i(s-\lambda)(\xi'^2+1)}\langle\begin{pmatrix}x|\tilde{U}^{(s)}|^4\tilde{W}^{(s)}(\lambda,.)\\ -x|\tilde{U}^{(s)}|^4\overline{\tilde{W}^{(s)}}(\lambda,.)\end{pmatrix},(e^{ix\xi'}+e^{-ix\xi'})\sigma_1\underline{e}\rangle d\lambda' d\xi d\xi'|\lesssim T^{-\frac{1}{2+}}\end{aligned}$$

$$\begin{aligned}|\sum_k\int_{-\infty}^{0}\int_{-\infty}^{0}\int_T^{\infty}t\int_t^{\infty}&\phi_{<s^{\epsilon}}(\xi)\phi_{<s^{\epsilon}}(\xi')\xi g(\xi,\xi')\\ &\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda}e^{-i(s-\lambda)(\xi^2+1)}\langle\begin{pmatrix}\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\tilde{W}^{(s)}(\lambda,.)\\ -\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{W}^{(s)}}(\lambda,.)\end{pmatrix},(e^{ix\xi}+e^{-ix\xi})\underline{e}\rangle d\lambda\\ &\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda'}e^{+i(s-\lambda)(\xi'^2+1)}\langle\begin{pmatrix}x|\tilde{U}^{(s)}|^4\tilde{V}^{(s)}(\lambda,.)\\ -x|\tilde{U}^{(s)}|^4\overline{\tilde{V}^{(s)}}(\lambda,.)\end{pmatrix},(e^{ix\xi'}+e^{-ix\xi'})\sigma_1\underline{e}\rangle d\lambda' d\xi d\xi'|\lesssim T^{-\frac{1}{2+}}\end{aligned}$$

$$\begin{aligned}|\sum_k\int_{-\infty}^{0}\int_{-\infty}^{0}\int_T^{\infty}t\int_t^{\infty}&\phi_{<s^{\epsilon}}(\xi)\phi_{<s^{\epsilon}}(\xi')\xi g(\xi,\xi')\\ &\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda}e^{-i(s-\lambda)(\xi^2+1)}\langle\begin{pmatrix}\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\tilde{V}^{(s)}(\lambda,.)\\ -\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4\overline{\tilde{V}^{(s)}}(\lambda,.)\end{pmatrix},(e^{ix\xi}+e^{-ix\xi})\underline{e}\rangle d\lambda\\ &\times\int_0^{\frac{s}{2}}\frac{1}{s-\lambda'}e^{+i(s-\lambda)(\xi'^2+1)}\langle\begin{pmatrix}x|\tilde{U}^{(s)}|^4\tilde{W}^{(s)}(\lambda,.)\\ -x|\tilde{U}^{(s)}|^4\overline{\tilde{W}^{(s)}}(\lambda,.)\end{pmatrix},(e^{ix\xi'}+e^{-ix\xi'})\sigma_1\underline{e}\rangle d\lambda' d\xi d\xi'|\lesssim T^{-\frac{1}{2+}},\end{aligned}$$

which together with (5.56) implies

$$|\sum_k \int_{-\infty}^0 \int_{-\infty}^0 \int_T^\infty t \int_t^\infty \phi_{<s^\epsilon}(\xi)\phi_{<s^\epsilon}(\xi')\xi g(\xi,\xi')$$
$$\times \int_0^{\frac{s}{2}} \frac{1}{s-\lambda} e^{-i(s-\lambda)(\xi^2+1)} \langle \begin{pmatrix} \phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4 \tilde{U}^{(s)}(\lambda,.) \\ -\phi_{\sim 2^k}(x)|\tilde{U}^{(s)}|^4 \overline{\tilde{U}^{(s)}}(\lambda,.) \end{pmatrix}, (e^{ix\xi}+e^{-ix\xi})\underline{e}\rangle d\lambda$$
$$\times \int_0^{\frac{s}{2}} \frac{1}{s-\lambda'} e^{+i(s-\lambda)(\xi'^2+1)} \langle \begin{pmatrix} x|\tilde{U}^{(s)}|^4 \tilde{U}^{(s)}(\lambda,.) \\ -x|\tilde{U}^{(s)}|^4 \overline{\tilde{U}^{(s)}}(\lambda,.) \end{pmatrix}, (e^{ix\xi'}+e^{-ix\xi'})\sigma_1\underline{e}\rangle d\lambda' d\xi d\xi'| \lesssim T^{-\frac{1}{2+}},$$

as desired.
The case $\max\{\lambda,\lambda'\} > \frac{s}{2}$ is more of the same. This concludes the proof of the estimate for the bilinear symplectic form. We have now also filled the gap in retrieving control over $||CU||_{L_x^2}$: while the ϕ in the expression $\langle \tilde{U}^2 - \overline{\tilde{U}}^2, \phi\rangle$ encountered there was time dependent, one easily checks that up to an error which leads to an absolutely integrable expression, one may replace this by a constant function. □

We are now also done with the proof of Proposition 5.4, since all terms arising upon substituting (3.41), (3.40) into $\int_T^\infty t\lambda_6(t)dt$ are controlled either by Lemma 5.6, Lemma 5.7, or else can be absolutely integrated. □

5.5. **Retrieving control over the modulation parameters.** We commence with $\beta_1 = \beta\nu - b_\infty\lambda_\infty^{-1}$, which is given by the righthand side of (3.25). Observe that schematically we have

$$E_2(\sigma) = -\langle N, \tilde{\xi}_2\rangle + \sum_{a=0,1} (\nu-1)^a \lambda_6(\sigma) + (\nu-1)(\sigma)\langle \begin{pmatrix} \tilde{U} \\ \overline{\tilde{U}} \end{pmatrix}_{dis}, \phi\rangle$$

Thus we recover the desired estimate for $\beta_1(s)$ upon using Proposition 5.4, if we also show that

$$|\lambda_\infty^{-1}(s)\int_s^T \lambda_\infty(\sigma)(\nu-1)(\sigma)\langle \begin{pmatrix} \tilde{U} \\ \overline{\tilde{U}} \end{pmatrix}_{dis}, \phi\rangle| \lesssim \langle s\rangle^{-\frac{3}{2}+\delta_1}$$

We state the

Lemma 5.8. *Let $\Gamma \in A_{[0,T)}^{(n)}$, as usual n sufficiently large. Then we have*

$$\int_s^T \lambda_\infty(\sigma)(\nu-1)(\sigma)\langle \begin{pmatrix} \tilde{U} \\ \overline{\tilde{U}} \end{pmatrix}_{dis}(\sigma), \phi\rangle| \lesssim \langle s\rangle^{-\frac{1}{2}+\delta_1}$$

Proof. This is proved as usual by integration by parts in t, and Duhamel-expanding $\begin{pmatrix} \tilde{U} \\ \overline{\tilde{U}} \end{pmatrix}_{dis}(\sigma)$. Thus we rewrite the expression as the sum of the terms

$$\int_{-\infty}^\infty \int_T^\infty t(\nu-1)(t)\int_0^t e^{i(t-\lambda)(\xi^2+1)} \mathcal{F}\begin{pmatrix} (1-e^{2i(\Psi-\Psi_\infty)_1(t)-2i(\Psi-\Psi_\infty)_1(\lambda)})\overline{\tilde{U}^{(t)}(\lambda,.)}\phi_0^4 \\ (-1+e^{2i(\Psi-\Psi_\infty)_1(t)-2i(\Psi-\Psi_\infty)_1(\lambda)})\tilde{U}^{(t)}(\lambda,.)\phi_0^4 \end{pmatrix}(\xi)\overline{\tilde{\mathcal{F}}\phi(\xi)}d\lambda dt d\xi$$

$$\int_{-\infty}^\infty \int_T^\infty t(\nu-1)(t)\int_0^t e^{i(t-\lambda)(\xi^2+1)} \mathcal{F}\begin{pmatrix} |\tilde{U}^{(t)}|^4(\lambda,.)\tilde{U}^{(t)}(\lambda,.) \\ -|\tilde{U}^{(t)}|^4(\lambda,.)\tilde{U}^{(t)}(\lambda,.) \end{pmatrix}(\xi)\overline{\tilde{\mathcal{F}}\phi(\xi)}d\lambda dt d\xi,$$

as well as faster decaying local terms which can be handled similarly to the first term. Let's look at the 2nd term here, the first being treated as in the proof of lemma 5.5 by integrations by parts and further Duhamel expansion. Perform an integration by parts, replacing this by the sum of suitable multiples of

$$A = \int_{-\infty}^\infty T(\nu-1)(T)\int_0^T e^{i(T-\lambda)(\xi^2+1)} \mathcal{F}\begin{pmatrix} |\tilde{U}^{(T)}|^4(\lambda,.)\tilde{U}^{(T)}(\lambda,.) \\ -|\tilde{U}^{(T)}|^4(\lambda,.)\tilde{U}^{(T)}(\lambda,.) \end{pmatrix}(\xi)\frac{\overline{\tilde{\mathcal{F}}\phi(\xi)}}{\xi^2+1}d\lambda dt d\xi,$$

$$B = \int_{-\infty}^\infty \int_T^\infty \frac{d}{dt}[t(\nu-1)(t)]\int_0^t e^{i(t-\lambda)(\xi^2+1)} \mathcal{F}\begin{pmatrix} |\tilde{U}^{(t)}|^4(\lambda,.)\tilde{U}^{(t)}(\lambda,.) \\ -|\tilde{U}^{(t)}|^4(\lambda,.)\tilde{U}^{(t)}(\lambda,.) \end{pmatrix}(\xi)\frac{\overline{\tilde{\mathcal{F}}\phi(\xi)}}{\xi^2+1}d\lambda dt d\xi,$$

$$C=\int_{-\infty}^{\infty}\int_{T}^{\infty}t(\nu-1)(t)\frac{d}{dt}[\lambda_{\infty}(\mu-\mu_{\infty})(t)]$$
$$\int_{0}^{t}e^{i(t-\lambda)(\xi^2+1)}\mathcal{F}\begin{pmatrix}|\tilde{U}^{(t)}|^4(\lambda,.)\partial_x\tilde{U}^{(t)}(\lambda,.)\\-|\tilde{U}^{(t)}|^4(\lambda,.)\partial_x\tilde{U}^{(t)}(\lambda,.)\end{pmatrix}(\xi)\frac{\overline{\tilde{\mathcal{F}}\phi(\xi)}}{\xi^2+1}d\lambda dt d\xi,$$

$$D=\int_{-\infty}^{\infty}\int_{T}^{\infty}t(\nu-1)(t)\frac{d}{dt}[\lambda_{\infty}(\mu-\mu_{\infty})(t)]$$
$$\int_{0}^{t}e^{i(t-\lambda)(\xi^2+1)}\mathcal{F}\begin{pmatrix}\partial_x[|\tilde{U}^{(t)}|^4](\lambda,.)\tilde{U}^{(t)}(\lambda,.)\\-\partial_x[|\tilde{U}^{(t)}|^4](\lambda,.)\tilde{U}^{(t)}(\lambda,.)\end{pmatrix}(\xi)\frac{\overline{\tilde{\mathcal{F}}\phi(\xi)}}{\xi^2+1}d\lambda dt d\xi,$$

$$E=\int_{-\infty}^{\infty}\int_{T}^{\infty}t(\nu-1)(t)\frac{d}{dt}[\Psi-\Psi_{\infty}]_1(t)\int_{0}^{t}e^{i(t-\lambda)(\xi^2+1)}\mathcal{F}\begin{pmatrix}|\tilde{U}^{(t)}|^4(\lambda,.)\tilde{U}^{(t)}(\lambda,.)\\|\tilde{U}^{(t)}|^4(\lambda,.)\tilde{U}^{(t)}(\lambda,.)\end{pmatrix}(\xi)\frac{\overline{\tilde{\mathcal{F}}\phi(\xi)}}{\xi^2+1}d\lambda dt d\xi,$$

$$F=\int_{-\infty}^{\infty}\int_{T}^{\infty}t(\nu-1)(t)\mathcal{F}\begin{pmatrix}|\tilde{U}|^4(t,.)\tilde{U}(t,.)\\-|\tilde{U}|^4(t,.)\tilde{U}(t,.)\end{pmatrix}(\xi)\frac{\overline{\tilde{\mathcal{F}}\phi(\xi)}}{\xi^2+1}dt d\xi,$$

Now, for A, repeat the proof of SLDE[81] to bound it by $\lesssim T^{\frac{1}{2}+\delta_1}T^{-\frac{3}{2}+\delta_3}$, better than what is needed. For B, use that $|\frac{d}{dt}[t(\nu-1)(t)]|\lesssim t^{-\frac{1}{2}+2\delta_1}$, see (3.32). C is handled analogously. For D, observe that[82]

$$\int_{-\infty}^{\infty}\int_{0}^{t}e^{i(t-\lambda)(\xi^2+1)}\mathcal{F}\begin{pmatrix}\partial_x[|\tilde{U}^{(t)}|^4](\lambda,.)\tilde{U}^{(t)}(\lambda,.)\\-\partial_x[|\tilde{U}^{(t)}|^4](\lambda,.)\tilde{U}^{(t)}(\lambda,.)\end{pmatrix}(\xi)\frac{\overline{\tilde{\mathcal{F}}\phi(\xi)}}{\xi^2+1}d\lambda d\xi$$
$$=\langle\int_{0}^{t}e^{i(t-\lambda)\mathcal{H}}\begin{pmatrix}\partial_x[|\tilde{U}^{(t)}|^4](\lambda,.)\tilde{U}^{(t)}(\lambda,.)\\-\partial_x[|\tilde{U}^{(t)}|^4](\lambda,.)\tilde{U}^{(t)}(\lambda,.)\end{pmatrix}_{dis}d\lambda,\mathcal{H}^{-1}\phi\rangle$$

Proceeding as in the proof of SLDE, i. e. using (5.3) as well as pseudo-conformal almost conservation, we can bound the preceding expression by $\lesssim\langle t\rangle^{-\frac{3}{2}+\delta_3}$. From here one proceeds as for C etc. Finally, F is more elementary, as we have

$$|\int_{-\infty}^{\infty}\mathcal{F}\begin{pmatrix}|\tilde{U}|^4(t,.)\tilde{U}(t,.)\\-|\tilde{U}|^4(t,.)\tilde{U}(t,.)\end{pmatrix}(\xi)\frac{\overline{\tilde{\mathcal{F}}\phi(\xi)}}{\xi^2+1}d\xi|=|\langle\begin{pmatrix}|\tilde{U}|^4(t,.)\tilde{U}(t,.)\\-|\tilde{U}|^4(t,.)\tilde{U}(t,.)\end{pmatrix}_{dis},\mathcal{H}^{-1}\phi\rangle|\lesssim t^{-\frac{9}{2}+\epsilon(\delta_2)}$$

□

It is now straightforward to retrieve the desired bound for $\beta_1(s)$ via (3.25), and from here one easily infers the desired bound for $\nu_1(s)=\nu(s)-1$, via (3.24).

Next, consider ω satisfying (3.26). From what we have established so far we can retrieve the bound

$$B(s)^{-1}=c\lambda_{\infty}^{-1}(s)+O(s^{-\frac{3}{2}+\delta_1})$$

From here, (3.26) and Lemma 5.8 one infers the existence of c_{∞} such that

$$|\omega-c_{\infty}\lambda_{\infty}^{-1}(s)|\lesssim s^{-\frac{3}{2}+\delta_1}$$

Moving on to μ satisfying (3.29), we deduce the existence of v_{∞},y_{∞} such that

$$|\mu(s)-\frac{2v_{\infty}s+y_{\infty}}{a_{\infty}+b_{\infty}s}|\lesssim s^{-\frac{3}{2}+\delta_1},$$

and furthermore we have $c_{\infty}=v_{\infty}a_{\infty}-\frac{b_{\infty}y_{\infty}}{2}$. Finally, one easily deduces the bound on $\gamma(s)-s$ from (3.31). To get the estimates specified in (3.32) on the derivatives $\dot{\nu}(s)$ etc., one simply differentiates the relations (3.25) etc, and uses the assumptions. This is elementary and hence omitted. To complete the proof of Theorem 5.1, we need to show that also alternating iterate formation with forming convex linear combinations sufficiently often leads to the desired bounds. To put this concisely, we claim:

[81]SLDE

[82]Of course, one should include the $\pm$ subscripts for $\mathcal{F}$, $\tilde{\mathcal{F}}$.

Proposition 5.9. *Assume we are given a tree of tuples on $[0,T)\times\mathbf{R}$ as follows:*

$$tuple_1=\sum_{l=1}^{m}\alpha_{1l}tuple_{2,l},\ \alpha_{1l}\geq 0\forall l,\ \sum_{l=1}^{m}\alpha_{1l}=1$$

$$tuple_{2,k_1}=T_A[\sum_{l=1}^{m_{k_1}}\alpha_{2k_1}tuple_{3,k_1 l}],\ 1\leq k_1\leq m,\ \alpha_{2k_1 l}\geq 0,\ \sum_l\alpha_{2k_1 l}=1,\ldots$$

$$tuple_{L,k_1k_2\ldots k_{L-1}}=T_A[\sum_{l=1}^{m_{k_1\ldots k_{L-1}}}\alpha_{Lk_1k_2\ldots k_{L-1}l}tuple_{L+1,k_1k_2\ldots k_{L-1}l},\ 1\leq k_1\leq m,1\leq k_2\leq m_{k_1}\ldots$$

Moreover, assume that

$$|||tuple_{i,k_1\ldots k_{i-1}}|||_{\tilde{S}^{N,K}}\leq R\delta,\ 1\leq i\leq L$$

Then there is some universal L and function $K(T)$ as before such that provided R is large enough in relation to $|||A|||$ and δ small enough in relation to R, we obtain

$$|||T_A[tuple_1]|||_{\tilde{S}^{N,K}}\leq\frac{R}{2}\delta,$$

irrespective of the numbers $m, m_{k_1}\ldots$ as well as $\alpha_{1l},\alpha_{2k_1l}$ etc.

Proof. This follows by basically the exact same proof as that given before. The only difference is that at each juncture where we iterate the equations, either for the radiation part U via Duhamel's equation, or the remaining parameters (root and modulation) via their corresponding ODDE's, we substiute a convex linear combination of the corresponding expressions. Given the fact that these expressions are always essentially[83] multilinear, the same estimates go through. □

Finally we have completed the proof of Theorem 4.5, up to the continuity assertion. To show the latter, *note that we are working on the finite time interval* $[0,T)$, whence continuity can be derived by using very crude bounds from linear theory. □

References

Agm. Agmon, S. *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators.* Mathematical Notes, 29. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982.

BerCaz. Berestycki, H., Cazenave, T. *Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires.* C. R. Acad. Sci. Paris Sér. I Math. 293 (1981), no. 9, 489–492.

B. Bourgain, J., *Global solutions of nonlinear Schroedinger equations*, Amer. Math. Soc. Colloquium Publ., vol. 46, 1999

BW. Bourgain, J., Wang, W., *Construction of blowup solutions for the nonlinear Schroedinger equation with critical nonlinearity*, Annali Scuola Ecole Normale Cl.(4) 25 (1997), no.1-2, 197-215

BusPer. Buslaev, V. S., Perelman, G. S. *Scattering for the nonlinear Schrödinger equation: states that are close to a soliton.* (Russian) Algebra i Analiz 4 (1992), no. 6, 63–102; translation in St. Petersburg Math. J. 4 (1993), no. 6, 1111–1142.

CazLio. Cazenave, T., Lions, P.-L. *Orbital stability of standing waves for some nonlinear Schrödinger equations.* Comm. Math. Phys. 85 (1982), 549–561

CriKis. Christ, M., Kiselev, A. *Maximal functions associated with filtrations,* J. Funct. Anal. 179 (2001), 409-425.

Cuc. Cuccagna, S. *Stabilization of solutions to nonlinear Schrödinger equations.* Comm. Pure Appl. Math. 54 (2001), no. 9, 1110–1145.

Flu. Flügge, S. *Practical quantum mechanics.* Reprinting in one volume of Vols. I, II. Springer-Verlag, New York-Heidelberg, 1974.

Froe. Froehlich, J., Gustafson, S., Jonsson, B. L. G., Sigal, I. M., *Solitary wave dynamics in an external potential.* Comm. Math. Phys. 250 (2004), no. 3, 613–642.

GolSch. Goldberg, M., Schlag, W. *Dispersive estimates for Schrödinger operators in dimensions one and three.* Comm. Math. Phys. 251 (2004), no. 1, 157–178.

Gri. Grillakis, M. *Analysis of the linearization around a critical point of an infinite dimensional Hamiltonian system.* Comm. Pure Appl. Math. 41 (1988), no. 6, 747–774.

Har. Hartman, P. *Ordinary differential equations.* Corrected reprint of the second (1982) edition. Classics in Applied Mathematics, 38. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.

[83]Note that the map T_A is highly nonlinear and not even convex. However, this does not affect this argument, as is easily verified.

Kat. Kato, T. *Wave operators and similarity for some non-selfadjoint operators.* Math. Ann. 162 (1965/1966), 258–279.

KriSch1. Krieger, J., Schlag, W. *Stable manifolds for all monic supercritical NLS in one dimension.*, preprint

KriSch2. Krieger, J., Schlag, W. *On the focusing critical semi-linear wave equation*, preprint

LiWig. Li, C., Wiggins, S. *Invariant manifolds and fibrations for perturbed nonlinear Schrdinger equations.* Applied Mathematical Sciences, 128. Springer-Verlag, New York, 1997.

MeRa. Merle, F., Raphael, P. *On universality of blow-up profile for L^2 critical nonlinear Schroedinger equation*, Invent. Math. 156(2004), no. 3, 565-672

Ra. Raphael, P., *Stability of the log-log bound for blow up solutions to the critical nonlinear Schroedinger equation*, Math. Ann. 331(2005), no.3, 577-609

Mur. Murata, M. *Asymptotic expansions in time for solutions of Schrödinger-type equations* J. Funct. Anal. 49 (1) (1982), 10–56.

Per2. Perelman, G. *On the formation of singularities in solutions of the critical nonlinear Schrödinger equation.* Ann. Henri Poincaré 2 (2001), no. 4, 605–673.

ReeSim4. Reed, M., Simon, B. *Methods of modern mathematical physics. IV.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1979.

RodSch. Rodnianski, I., Schlag, W. *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials.* Invent. Math. 155 (2004), no. 3, 451–513.

Sch. Schlag, W. *Stable manifolds for an orbitally unstable NLS.* preprint, 2004.

Sch2. Schlag, W. *Dispersive estimates for Schrödinger operators: A survey.* preprint 2004, to appear in proceedings of the conference "Workshop on Aspects of nonlinear PDE's", IAS Princeton.

SogSmi. Smith, H., Sogge, C. *Global Strichartz estimates for nontrapping perturbations of the Laplacian.* Comm. Partial Differential Equations 25 (2000), no. 11-12, 2171–2183.

SofWei1. Soffer, A., Weinstein, M. *Multichannel nonlinear scattering for nonintegrable equations.* Comm. Math. Phys. 133 (1990), 119–146

SofWei2. Soffer, A., Weinstein, M. *Multichannel nonlinear scattering, II. The case of anysotropic potentials and data.* J. Diff. Eq. 98 (1992), 376–390

St. Stein, E., *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993

TsaYau. Tsai, T.-P., Yau, H.-T. *Stable directions for excited states of nonlinear Schrödinger equations.* Comm. Partial Differential Equations 27 (2002), no. 11-12, 2363–2402.

Wei1. Weinstein, Michael I. *Modulational stability of ground states of nonlinear Schrödinger equations.* SIAM J. Math. Anal. 16 (1985), no. 3, 472–491.

Wei2. Weinstein, Michael I. *Lyapunov stability of ground states of nonlinear dispersive evolution equations.* Comm. Pure Appl. Math. 39 (1986), no. 1, 51–67.

Harvard University, Dept. of Mathematics, Science Center, 1 Oxford Street, Cambridge, MA 02138, USA
E-mail address: jkrieger@math.harvard.edu

Department of Mathematics, University of Chicago, 5734 South University Avenue, Chicago, Il 60637, USA
E-mail address: schlag@math.uchicago.edu